Zeitschrift: IABSE publications = Mémoires AIPC = IVBH Abhandlungen

Band: 16 (1956)

Artikel: On impact accompanied by fatigue

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DOI: https://doi.org/10.5169/seals-15072

On Impact Accompanied by Fatigue[1])

Choc accompagné de fatigue

Ermüdungsfestigkeit nach Stoßbeanspruchung

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Introduction

In any structure excited or driven by a system of time-dependent forces the response of the structure may be considered to fall in one of three categories: a) Oscillatory motion of constant frequency, b) Transient motion, or c) Oscillatory motion not of constant frequency. The term "vibration" is usually applied to any periodically varying motion which may be either steady-state or transient in nature. Such steady-state motions may consist of one or more frequencies with the motion at each frequency being harmonic. Consequently steady-state vibration may be completely defined by specifying the frequencies and corresponding amplitudes of motion.

For the purposes of this paper a rapidly applied transient motion will be termed a shock. Such motion is usually not sinusoidal in nature and is applied over a small but finite period of time. If the equilibrium of a structure is disrupted by a suddenly applied force or increment of force, or by a sudden change in the magnitude or direction of velocity a state of shock is said to exist. Unfortunately it is not possible to define a shock motion merely by stating numerical values of the usual force or motion parameters. A complete definition is offered only by the displacement-time, velocity-time, or acceleration-time records of the motion. When any elastic structure is subjected to a transient disturbance the response of the system is influenced by the ratios

[1]) For presentation at the Fifth Congress of the International Association for Bridge and Structural Engineering, Lisbon, Portugal, June 25—July 2, 1956.

of its natural frequencies of vibration to the frequency components of the disturbance.

During recent years problems involving transient disturbances in the form of single or repeated impact loadings on structures have become of ever-increasing importance to engineers. To cite but a few examples we might mention the design of steel structures housing certain large pieces of manufacturing equipment, such as drop forging machines and punch presses. Structures housing such machinery are subject to large shock-type forces resulting from the operation of these machines. In certain cases these shocks are applied a sufficiently large number of times in the life of the structure that it becomes necessary to consider the possibility of a fatigue failure. Many other examples of shock loadings on elastic structures occur in the fields of aircraft design and naval architecture. In certain cases empirical relations exist that indicate the addition of a certain percentage of the live-load stress to account for impact effects. Obviously any such formula attempting to describe a complex dynamic phenomena without introducing a time-history of the transient disturbance is at best approximate. Frequently the effects of the transient disturbances applied to the structure may best be described as a series of shocks acting on the structure.

Clearly there exists the need for some measure of the damaging potential of a single shock load. In an effort to obtain simplified design and analysis considerations it is possible that an attempt might be made to transform the dynamic load into an equivalent static load, i.e. a load which if applied statically will produce identical strains, deflections, and stresses. Unfortunately conventional static considerations deal with the two variables of load and resistance, whereas dynamic analysis deals with the variables of intensity and duration of load, mass, strength, and permissible plastic strains. Consequently, a simple equivalent static load can be found only for the most elementary structures. Even then important approximations must be made in regard to the nature of the loading.

Previous analyses of the effect of shock motions upon structures (1 through [5])[2]) have indicated the desirability of regarding the behavior of the system as a measure of the shock intensity, rather than attempting to use the conventional idea of force magnitude to describe the shock. This is the approach that will be followed in the present paper, both with regard to a single shock and also a series of applied shock loadings.

Modern electronic transducers and recording equipment make it possible to obtain a time history of any motion characteristic, say acceleration, of a selected point or points on a structure. In the case of a steel building for example, it would be logical to use as transducers either seismicmass type or crystal-type accelerometers attached to various points on the vertical steel

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[2]) Numbers in parentheses refer to the bibliography at the end of the paper.

columns or horizontal girders of the structure. The output of these accelerometers would be recorded on a recording oscillograph so that a complete time history of the motion at the point under investigation would be obtained. In its most general form this record would be non-periodic and the analysis to follow applies to either a non-periodic or a periodic time history.

The objective of this paper is to present an exposition of a new method for interpreting experimentally determined acceleration-time records obtained at a given point on any elastic structure. The method also enables the investigator to take fatigue effects into account. This point is of importance in structures housing heavy equipment as mentioned previously and also in railway bridges that are repeatedly subject to impact loadings during their lifetime. It is to be emphasized that the following techniques apply to any linear elastic structure subject to impact loading, or even repeated impacts where fatigue is of importance.

In the following discussion attention will be focused mainly on simple systems of one-degree-of-freedom which are subject to shocks not severe enough to exceed the elastic limit of the material. Actually all physical structures possess many-degrees-of-freedom but frequently one of these is so predominant that it determines the behavior of the system for all practical purposes. For any elastic structure subject to statically applied loads there exists a unique deflection profile. When these same loads are applied over a very short period of time in the form of a shock loading, the structure may be said to respond as a system of one-degree-of-freedom if the instantaneous deflection profile in the extreme position is geometrically similar to that existing under static load. To a first approximation it would appear reasonable to treat the problem of determining the dynamic response of many structures as equivalent to determining the time history of the motion of a lumped single-degree-of-freedom system. In those cases where this simplification is unsatisfactory, one may recognize that any complex system can be formed from the superposition of a series of single-degree-of-freedom systems, corresponding to the normal modes of vibration of the complex system. The response of the complex system may be determined by superposition of the responses of the component systems although it is difficult to take account of phase relationships. The assumption that the maximum responses of all of the single-degree-of-freedom systems comprising the complex system occur simultaneously will overestimate the total response, hence will err in a conservative manner.

Severity of a Single Shock

Response of a spring supported mass to shock. Let us consider first the most elementary type of dynamic system, namely a spring-supported mass which is designated as a single-degree-of-freedom system. The response of this simple system to shock excitation may be determined by either of several methods. If the variation of the applied shock excitation is known as some function of time then the response of the system to such a shock load can be obtained by use of Duhamel's Integral [6]. By this technique it is possible to determine the displacement-time relation for the mass once the natural frequency of the system is known. More frequently the analytical expression for the variation of excitation with time is not known but instead some graphical representation of this variation is available. This usually assumes the form of an acceleration-time plot obtained from any one of several commonly used types of accelerometers. In this case two possibilities exist for determining the response of the system. The technique that suggests itself first is a graphical evaluation of Duhamel's Integral [7]. The second technique is to employ the so-called Biot torsion pendulum [8].

Descriptions of the behavior of mechanical systems subject to transient forces have been presented by several investigators employing the so-called response spectrum approach [1, 4, 5]. Such a response spectrum indicates the relationship between some maximum response parameter of a single-degree-of-freedom system and the natural frequency of the undamped system. The response parameter selected could be a displacement, a velocity, or an acceleration; the plot of the response parameter against the natural frequency of the single-degree-of-freedom system is termed the response spectrum.

Methods of finding the response spectrum. The problem of obtaining the response spectrum when the force-time relationship is known analytically may be solved by use of Duhamel's Integral [6]. In the event that it becomes necessary to resort to a numerical evaluation of this integral it would appear that electronic computer techniques would be appropriate. One possibility is to use any one of a number of commercially available analog computers whose electrical elements carry out the mathematical operations indicated by the differential equation governing the behavior of the single-degree-of-freedom system. This approach has been carried out successfully by CREDE, GERTEL, and CAVANAUGH [2]. Another possibility is to set up a direct mechanical-electrical analogy by constructing an electric circuit whose behavior is described by the same differential equations that govern the behavior of the mechanical system.

It is also possible to obtain the response spectrum by a more direct measurement. This may be accomplished by means of a so-called reed gage [9]. This instrument consists of sets of steel reeds of different natural frequencies fastened as cantilevers in a heavy frame. The frame is attached to the structure whose motions are being investigated. The reed deflections which are recorded during a shock motion produce a record which is analyzed for the shock motion characteristics.

The shock spectrum. The reed gage may be used to obtain a quantitative measure of the effects of a shock on a structure by mounting the gage on the structure and then applying some excitation to the frame of the gage. The maximum deflections of the reeds relative to the frame are recorded. If each reed is considered to be a single-degree-of-freedom system then the static deflection δ_s due to gravity alone is related to the natural frequency f_n of that reed by the equation

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_s}}.$$

If X denotes the deflection of the extremity of a reed relative to the frame of the instrument then we may form the ratio [1]

$$\frac{X}{\delta_s} = N$$

where N is termed the equivalent static acceleration. This equivalent static acceleration is the gradually applied acceleration, expressed as a multiple of the acceleration of gravity, to which the reed must be subjected in order to produce the same deflection of the tip as was produced by the applied shock. Successive values of N may then be plotted against the lowest natural frequency of the reed. This plot of N vs. f_n is defined to be the shock spectrum of the motion to which the reed was subjected. Consequently it is a special case of the response spectrum wherein the acceleration is regarded as the response parameter of the system. The shock spectrum is considered to be an index of the damaging capacity of a shock.

Consequently the spectrum criterion for the severity of shock is based upon the hypothesis that the estimate of shock severity should not be based upon any directly measurable characteristic of the shock record but instead should be based upon the response of a single-degree-of-freedom mass-spring system to the shock load. It is to be noted that the usual method of depicting that response is to plot the *maximum* response acceleration against the natural frequency for each degree of damping, and then to draw an envelope that includes all the individual curves corresponding to the various degrees of damping.

However, in certain applications it is desirable to consider the effects of those accelerations that are smaller than the maximum. In particular, if there are a large number of smaller accelerations associated with the maximum then the maximum alone would not be a reliable representation of the acceleration response record. Also, it is desirable to know how much information is lost in restricting the shock spectrum to the maximum response accelerations, even in those cases when one is willing to accept some approximations in the analysis.

The first objective of this paper will be to introduce a new approach to the shock spectrum concept which will enable the investigator to account for accelerations other than the maximum. To this end it will be necessary to briefly examine certain fatigue characteristics of metals.

Fatigue Properties of Metals

The idealized S-N curve. The fatigue life N of a metal at a given stress level, S, is by definition the number of reversible cycles of stressing, normally with a zero mean stress, where the maximum and minimum stresses are equal, respectively, to the positive and negative values of the given stress level, and after which the metal exhibits some agreed upon symptoms of failure. The endurance limit, E, is by definition that stress level below which a specimen of the metal will not fail in reversible stressing no matter how large the number of cycles becomes, but above which the specimen will fail after a finite number of cycles.

Fatigue investigations have indicated that there exists a considerable statistical variability in the fatigue properties of metals as characterized by the endurance limit. Also, different metals of course have different values of endurance limit. Lastly, size, shape, and various metallurgical factors influence the fatigue characteristics. Nevertheless, to a first approximation it is possible to present an idealized representation of fatigue characteristics of metals in the form of a simplified S-N curve, i. e. a plot depicting the variation of stress level with the number of cycles of reversible stress. The idealized S-N curve that will be used in this consideration is shown in figure 1. Here, the tensile strength of the material is denoted by S_t. The ordinate of 0.4 corresponding to the lower horizontal line is typical for steel and aluminium. The equation of the slanting portion of this plot is readily found to be

$$\frac{S}{S_t} = 0.128\,(9.88 - \log_{10} N). \tag{1}$$

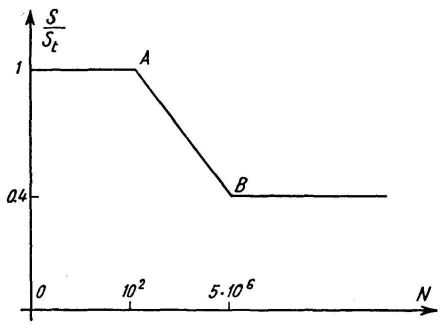

Fig. 1

Theories of fatigue damage. There exist many theories concerning the mechanism of fatigue damage. All of them agree that the fatigue damage is a function of the cycle ratio, i. e. the ratio of the number of cycles to which the specimen has been subjected to the total number of cycles representing its fatigue life. The various theories differ in regard to the type of functional relationship that exists between these quantities.

One of the simplest of these theories is due to MINER [10]. It hypothesizes that fatigue damage is a function of the cycle ratio only, and that this function is simply the equality of fatigue damage and cycle ratio. In equation form this theory states that

$$D = R$$

where D is the degree of damage and R is the cycle ratio. According to this hypothesis the damage incurred by a specimen due to a certain cycle ratio is independent of both the stress level at which that cycle ratio has been run and the position of that cycle relative to the total fatigue life of the specimen. Several more intricate theories introduce other functional relationships between D and R. However, for the purpose of this paper, it will be sufficient to consider only Miner's theory, although any other fatigue damage theory could be used in conjunction with the concepts to be presented in the following sections.

The Generalized Shock Spectrum

It is now possible to define and derive an expression for a new quantity, to be termed the "fatigue equivalent acceleration". The fatigue equivalent acceleration of some acceleration record is defined to be that acceleration level which will produce in one cycle an amount of fatigue damage equal to the cumulative damage produced by all the acceleration levels present in the given acceleration record. The conventional concept of the shock spectrum is based upon the premise of an entirely linear elastic system. Hence the stress in the spring of the single-degree-of-freedom mass-spring system tends to be proportional to the acceleration of the mass. Thus the stress ordinate in the S-N curve can be replaced by an acceleration ordinate multiplied by some scale factor. Consequently, the equation of the slanting line in the corresponding A-N curve is

$$\frac{A}{A_t} = 0.128\, c\,(9.88 - \log_{10} N) \tag{2}$$

where A is the acceleration level, A_t is the greatest acceleration level corresponding to S_t and c is a scale factor.

On the basis of Miner's fatigue damage theory it is evident that the amount of fatigue damage due to one cycle of reversible stressing corresponding to some acceleration level A is equal to $1/N_A$ where N_A is the number of cycles that corresponds to A in the A-N curve. Let d_A represent the amount of damage due to one cycle at the acceleration level A. Then, we have from equation (2)

$$d_A = \frac{1}{N_A} = 10^{-9.88} \cdot 10^{7.88\left(\frac{A}{A_t}\right)}. \tag{3}$$

If r denotes the ratio of the fatigue damage due to one cycle at some acceleration level A to the damage due to one cycle at some reference acceleration level A_{max}, then

$$r = \frac{d_A}{d_{A_{max}}} = 10^{7.88\left(\frac{A}{A_t} - \frac{A_{max}}{A_t}\right)}. \tag{4}$$

The maximum acceleration A_{max} is available from the acceleration-time record, but the magnitude of A_t is usually unknown. However, a conservative estimate of the ratio A_{max}/A_t is about 0.7 to 0.8. This assumption simplifies equation (4) to the following approximate form:

$$r = 10^{6\left(\frac{A}{A_{max}} - 1\right)}. \tag{5}$$

The assumed value of A_{max}/A_t stated above is equivalent to assuming that the maximum recorded acceleration corresponds to a stress level about equal to twice the endurance limit. Consequently any acceleration less than $A_{max}/2$ should produce no fatigue damage. Consequently we may replace equation (5) by the relations

$$r = \begin{cases} 10^{6\left(\frac{A}{A_{max}} - 1\right)} & \text{for} \quad A_{max}/2 < A \leqq A_{max} \\ 0 & \text{for} \quad 0 \leqq A \leqq A_{max}/2. \end{cases} \tag{6}$$

If equation (6) is now successively applied to all the acceleration levels in an acceleration record it is possible to evaluate the cumulative damage of the whole acceleration record, designated as r_c, in terms of the damage due to one cycle at the level of the maximum acceleration. This may be written in the form

$$r_c = \sum_{i=1}^{k} n_i \cdot 10^{6\left(\frac{A_i}{A_{max}} - 1\right)} \tag{7}$$

where $A_{max} = A_1 > A_2 > \cdots > A_k = A_{max}/2$ and n_i = number of cycles at level of A_i.

Equation (6) now indicates the acceleration level A_{eq} which will produce in one cycle fatigue damage equal to r_c as given by equation (7). The resulting expression for the fatigue equivalent acceleration is

$$A_{eq} = A_{max}\left[1 + \frac{1}{6}\log \sum_{i=1}^{k} n_i \cdot 10^{6\left(\frac{A_i}{A_{max}} - 1\right)}\right]. \tag{8}$$

Consequently, equation (8) indicates the equivalent acceleration level of a complete acceleration-time record. If the equivalent acceleration of each response record is plotted against the natural frequency of the mass-spring system giving the response, the resulting curve could be termed the "Generalized Shock Spectrum".

Response Surfaces and Shock Evaluation

An existing response surface. A recent investigation (2) seeking to extend the shock spectrum concept to account for certain fatigue effects advanced the hypothesis that the ability of an elastic member to withstand transient vibration may be determined by counting the number of cycles at each acceleration level embodied in the response and applying any one of several commonly accepted fatigue damage theories. The three parameters defining the response of the member are the natural frequency f_n of the element, the acceleration $\ddot{y}_0$ corresponding to the response, and the number of cycles n at each response acceleration amplitude. In an effort to depict the response acceleration of many elements having different natural frequencies, the excitation being a single record of acceleration as a function of time, the plot shown in figure 2 was proposed. This may be termed a response surface. The purpose of this representation was to describe the response of many systems having different natural frequencies to a single input acceleration.

Any dynamic system may be considered to consist of an assembly of component systems, each with its own characteristics. If each component system may be assumed to be a single-degree-of-freedom system with linear elasticity and damping, then each of these systems may be defined in terms of its natural frequency and damping. Consequently, the characteristics of the entire composite system are defined. The natural frequency is usually expressed in cycles per second and the damping parameter is often expressed in terms of a dimensionless quantity Q which indicates the maximum transmissibility at resonance during steady-state vibration.

Reference 2 advanced the theory that each shock is defined by a series of response surfaces of the type illustrated in figure 2. To each value of the damping parameter Q there would correspond a unique response surface. For a constant value of Q the general level of the response surfaces corresponding to various shock motions is said to indicate the relative severity of the respective motions.

The intent of Reference 2 was to present a three-dimensional representation of the shock spectrum, the third axis being introduced in an effort to give consideration to the number of occurrences at each acceleration level. Actually, as mentioned there, this particular response surface is perhaps better

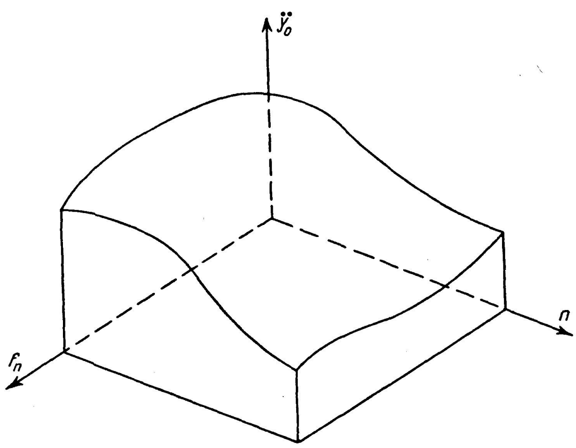

Fig. 2

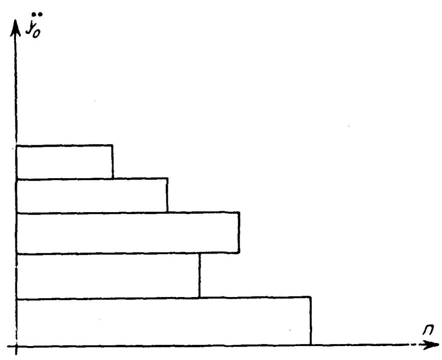

Fig. 3. Number of Occurrences (n) of response. Acceleration Amplitude

suited to qualitative rather than quantitative presentation. The authors contended that data may be presented more conveniently in two-dimensional plots of response acceleration vs. number of occurrences, as shown in figure 3, there being one such plot for each value of natural frequency f_n and damping Q of interest. The response surface of figure 2 is obviously a composite of a number of block diagrams of the type shown in figure 3 for various values of natural frequency f_n but for a constant value of the damping parameter Q. That is, each plane parallel to the $\ddot{y}_0 - n$ plane represents a block diagram analogous to that shown in figure 3 for a particular value of f_n.

However, the trace of this response surface on a plane parallel to the $\ddot{y}_0 - f_n$ plane does not represent the usual shock spectrum, even though its ordinates and abscissas have the same dimensions as do the corresponding axes for a conventional shock spectrum. This trace merely represents the variation of response acceleration with frequency for all responses occurring a constant, say n_1 times. Such a trace would not appear to be amenable to any physical interpretation. An even more serious characteristic of the response surface shown in figure 2 is the fact that it is represented there as a single-valued surface, i.e. to each pair of values of (f_n, n) there corresponds one and only one value of response acceleration $\ddot{y}_0$. However, there is no reason why all response accelerations of magnitudes less than the maximum (for a given f_n and Q) must occur a non-zero number of times. With reference to figure 3, it is apparent that one or more response accelerations of magnitudes less than the maximum may not occur at all, in which case the horizontal blocks representing them are of zero horizontal length. In that event the trace of the surface on the $\ddot{y}_0 - f_n$ plane is multiple-valued and the "general level" of the response surface has little or no meaning. Further, these traces on the $\ddot{y}_0 - f_n$ plane corresponding to response accelerations other than the maximum cannot be interpreted in the light of the usual shock spectrum concept. Consequently, it would appear that the trace of the response surface on the $\ddot{y}_0 - f_n$ plane corresponds to the shock spectrum only for those motions where the response surface is single-valued.

A proposed response surface. Since the response surface indicated by figure 2 is not considered to be an entirely satisfactory representation of the severity of a shock motion, a different type of response surface is presented in this report. Again, three orthogonal axes are employed. The first is an axis representing the natural frequency f_n as in figure 2. The second again represents the response acceleration of the structure. However, rather than employing the maximum response acceleration for a given number of occurrences, as was done in figure 2, the present acceleration axis corresponds to the fatigue-equivalent acceleration considered previously. In this manner it is possible to use only one co-ordinate axis to consider not only the maximum response acceleration but also acceleration levels lower than the maximum together with an indication of the number of occurrences of each acceleration level.

In figure 2, it was necessary to employ two coordinate axes to represent this same information, whereas by use of the present response surface this may be represented by means on only one axis. This leaves the third axis still to be selected and the damping factor Q may be chosen as the third coordinate. Thus in contrast to figure 2 which indicates a response surface for only a single value of the damping parameter Q it is possible to employ the present response surface to indicate dynamic characteristics for all values of frequency and damping of interest. Thus a single response surface represents the shock motion rather than a series of such surfaces each corresponding to a particular value of damping. Also, it is to be noted in the present response surface that it is not possible for the coordinate corresponding to the response acceleration to be multiple-valued as was the case in figure 2. Lastly, the trace of this response surface on a plane perpendicular to the Q-axis represents the shock spectrum for that particular value of damping. This was not the case in figure 2. In this report, the hypothesis is presented that the general level of the proposed response surface is an index of the severity of shock motion.

General Theory of Response Surfaces

Some definitions. In general, any functional relationship of the form

$$\eta = \eta\,(x_1, x_2 \ldots x_k)$$

where η denotes the response associated with certain values of the independent variables $x_1, x_2, \ldots x_k$ may be considered to constitute a *response surface.* In many investigations in the physical sciences the experimenter is interested in determining the peak response corresponding to a certain set of values of the x_i. In the most elementary case the response will be a function of only one variable and it will be convenient to plot the functional relationship by considering the response to be the ordinate and the value of the independent variable to be the abscissa.

If the response is considered to depend upon two independent variables x_1 and x_2 the investigator might perhaps be tempted to generalize by assuming that the response function could be represented by a surface having the contours shown in figure 4.

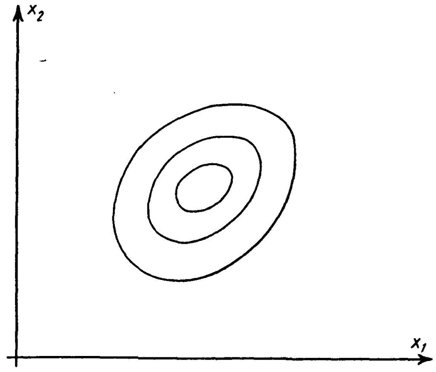

Fig. 4. Contours of constant response

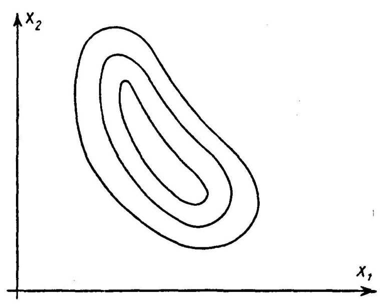

Fig. 5. Contours of constant response

However, actual response surfaces frequently are elongated or attenuated in the neighborhood of maxima as shown in figure 5 or they may have the appearance indicated by a ridge system as shown in figure 6. The situations illustrated in figures 5 and 6 are said to exhibit *factor dependence* since the response function for one factor is not independent of the levels of the other factors.

In investigations of response surfaces the topic of factor dependence is of considerable importance. For example, in the situation illustrated in figure 6 there exists not a point-type maximum but a series of points lying along the crest of a ridge and all corresponding to the same maximum response. Also, as will be discussed in greater detail later, the existence of factor dependence may possibly lead to a better understanding of the basic mechanism of the physical system that corresponds to the response surface.

The foregoing discussion pertains only to those systems characterized by two independent variables x_1 and x_2. The response corresponding to such a system may be illustrated either by means of a three dimensional diagram in which two dimensions are used to represent the variables x_1 and x_2 and the third to represent the response η, or by means of two dimensional diagrams such as those shown in figures 5 and 6 wherein the response is represented by contour lines.

In the case of a system characterized by three independent variables x_1, x_2, and x_3 the corresponding response may be depicted by means of the type of plot shown in figure 7 wherein the contour surfaces represent constant values of the response.

General methods for exploring the response surface. Frequently the experimenter is interested in determining the maximum response of the system under investigation. The customary procedure is to carry out a series of tests varying each of the several variables involved and in each instance determining the response of the system for the values of the variables selected. In order to

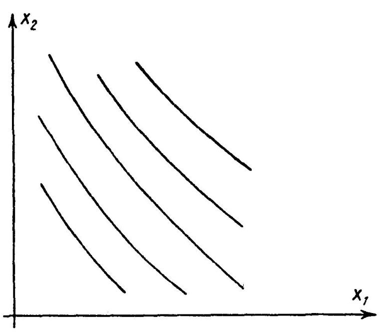

Fig. 6. Contours of constant response

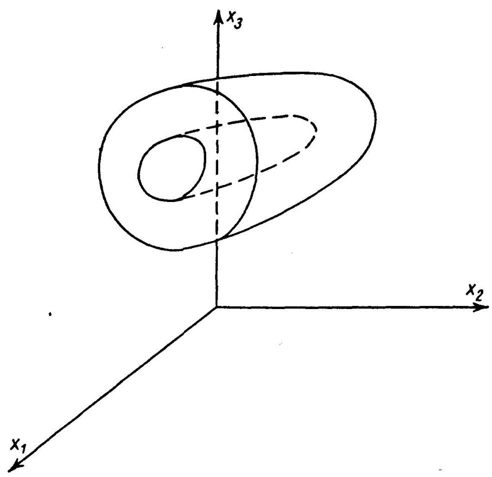

Fig. 7. Contours of constant response

determine the values of the variables corresponding to maximum response it is logical to begin by fitting a first degree equation representing a plane to the experimental data. The direction in which this plane slopes indicates the approximate location of the maximum response. That is, by proceeding up the plane in the direction of greatest slope one is led approximately to the maximum response. This is known as the method of *steepest ascent* [11, 12]. In the neighborhood of the indicated maximum response it would now be possible to fit a second degree equation to the experimental data. This would indicate more closely the coordinates of the point of maximum response. If it appeared that the second degree equation was still a rather poor fit to the experimental data it would be logical to proceed with the fitting of a third degree equation. For most physical processes this last step would be unnecessary.

The investigation of the response surface by means of a fitted second or third degree equation implies that an equation of the form

$$\begin{aligned} \eta = \beta_0 + (\beta_1 x_1 + \beta_2 x_2) + (\beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{12} x_1 x_2) \\ + (\beta_{111} x_1^3 + \beta_{222} x_2^3 + \beta_{112} x_1^2 x^2 + \beta_{122} x_1 x_2^2) + \cdots \end{aligned} \tag{9}$$

may be fitted to the experimental data. This is equivalent to stating that the function may be represented by its Taylor's series over the range of variables under investigation. Inspection of the above equation indicates that if two factors are involved in the problem the number of experimentally determined values of response needed to fit a plane to the data is 3, to fit a second degree equation (6), and to fit a third degree equation (10). In each case the equation may be fitted by the method of least squares. The number of experimentally determined response values cannot be less than each of these numbers and it usually exceeds the minimum.

It is of course desirable to have some knowledge of the goodness of fit of the equation fitted to the experimentally determined responses. To this end it is desirable to repeat a number of the determinations of response at certain selected points and thus determine the experimental error variance. The variance in this case is defined to be the square of the standard deviation of the response measurements taken at a particular point and using as the mean the mean of all of those response values at that point. Knowing this variance, an estimate of the goodness of fit may be found by certain statistical techniques discussed by Box [11].

Reduction of the fitted second degree equation to canonical form. As stated previously the general form of the second degree equation for two independent variables that is fitted to the experimentally determined response values is

$$\eta = b_0 + b_1 x_1 + b_2 x_2 + b_{11} x_1^2 + b_{22} x_2^2 + b_{12} x_1 x_2 \tag{10}$$

By the usual methods of analytic geometry it is possible to simultaneously translate the origin of the $x_1 - x_2$ coordinate system as well as to rotate the axes and thus reduce the above equation to the form

$$\eta - \eta_s = B_{11} X_1^2 + B_{22} X_2^2 \tag{11}$$

where η_s is the response predicted by the fitted equation. This is the so-called canonical form of the fitted equation. It will always be possible to reduce the general equation to one containing only the quadratic effects B_{11} and B_{22}. Only two possible geometric interpretations of equation (11) exist: a) If B_{11} and B_{22} are of like algebraic sign the response may be represented by the elliptical contours shown in figure 8, and b) if B_{11} and B_{22} are of opposite algebraic sign the response may be represented by the saddle-like surface shown in figure 9. For either of the above cases if one of the coefficients B_{11} or B_{22} is small in magnitude compared to the other then the response surface is elongated or attenuated along the axis corresponding to the smaller coefficient. It is to be noted that various special cases of the contours of figures 8 and 9 exist, occasionally taking the form of parallel straight lines. For example if $B_{22} = 0$ we have the contours depicted in figure 10.

Evidently the canonical form of the fitted equation possesses the desirable characteristic of simplicity. However, the reason for introducing the canonical form is based upon a much more important feature of the equation than mere simplicity. This feature, stated briefly, is that a study of the canonical form may suggest a new theory of behavior of the physical system. Thus, after re-writing the canonical equation in terms of the original variables, some new grouping or combination of these quantities may manifest itself. This could

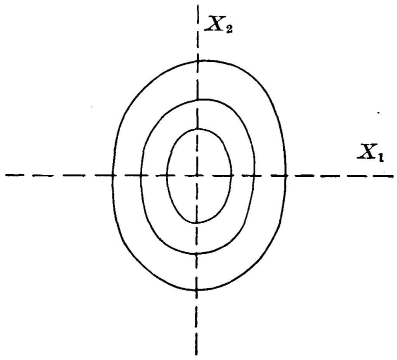

Fig. 8. Contours of constant response

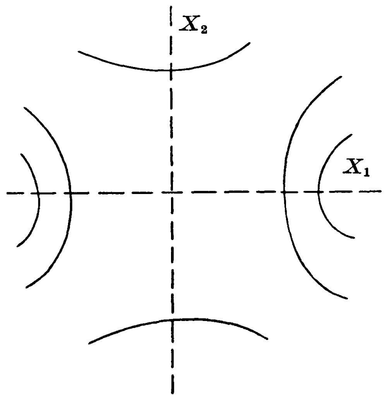

Fig. 9. Contours of constant response

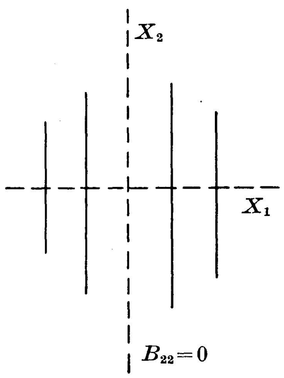

Fig. 10. Contours of constant response

perhaps appear in the form of a sum, or difference, or the ratio of the two original variables (11). Further, the relative magnitudes of the coefficients in the canonical equation indicate the relative importance of the canonical variables in describing the function over the region of interest.

Although it is not possible to draw any conclusions on the basis of the literal coefficients appearing in equation (11) it is possible in any given numerical situation that use of the canonical form of the fitted equation will lead to a valuable indication of the mechanism of behavior of the system.

Some General Observations

The fitted equation and its canonical form corresponding to some experimentally determined response enables the investigator to examine the behavior of the physical system over that range of variables corresponding to the equation. This examination may indicate new significant parameters of the system. It will also indicate the relative importance of the various canonical variables.

The above statements regarding response surfaces and canonical variables apply to many physical systems. Specific conclusions regarding shock phenomena may be drawn only in particular cases employing data from which it is possible to obtain numerical coefficients in both the fitted equation and also its canonical form.

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Summary

An index of the damaging capacity of an impact load on a structure is presented. Further, this concept is extended so as to account for fatigue damage resulting from multiple applications of the impact load. The concepts are general, applying to any linear elastic system. Lastly, a technique is presented for analyzing impact data so as to investigate the basic mechanism of behavior of the structure.

Résumé

On présente un indice de la capacité de destruction d'une charge de choc. En outre, on étend ce principe de façon à inclure la destruction par fatigue résultant de multiples applications de charges de choc. Ces principes sont généraux, pouvant être appliquées à n'importe quel système elastique linéaire. A la fin on présente une technique pour le calcul de la caractéristique du choc ayant pour but la recherche du mécanisme de base du comportement de la structure.

Zusammenfassung

Es wird eine Zusammenstellung der gefährlichen Auswirkungen einer stoßweise aufgebrachten Last gegeben.

Im weitern umfaßt der Bericht die gefährlichen Auswirkungen, die aus der wiederholten stoßweisen Belastung entstehen.

Die Ausführungen sind allgemein gehalten und können auf irgend ein lineares elastisches System angewandt werden.

Am Schluß wird ein Verfahren erläutert, um die Wirkung von stoßweise aufgebrachten Lasten zu untersuchen sowie um den grundlegenden Mechanismus über das strukturelle Verhalten zu erforschen.