Zeitschrift: IABSE proceedings = Mémoires AIPC = IVBH Abhandlungen

Band: 4 (1980)

Heft: P-32: Bending theory of skew-anisotropic plates

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DOI: https://doi.org/10.5169/seals-34957

Bending Theory of Skew-Anisotropic Plates

Théorie de flexion des plaques anisotropes biaises

Die Biegetheorie der schiefwinklig-anisotropen Platten

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SUMMARY

This paper puts forward the new theory of structural skew-anisotropic plates. Firstly, one has to determine the conditions of restraint in the direction of the plate plane. If there is no restraint in the direction of the plate plane, then the stiffness coefficient will be calculated according to eq. (28) of this paper, whereas, in the case of fixed restraint in the direction of the plate plane, the stiffness coefficient will be calculated according to eq. (24) of this paper.

RÉSUMÉ

Cet article présente la nouvelle théorie sur les structures composées de plaques anisotropes biaises. Il faut d'abord déterminer les conditions de limitation pouvant intervenir dans le plan de la plaque. S'il n'y a pas de limitation, on peut calculer le coefficient de rigidité selon l'équation (28) de cet article; si une limitation déterminée entre en ligne de compte, le coefficient de rigidité peut être calculé selon l'équation (24).

ZUSAMMENFASSUNG

In diesem Beitrag wird die neue Theorie der Konstruktion von schiefwinklig-anisotropen Platten aufgestellt. Es werden in erster Linie die Beschränkungsbedingungen der Platte in der ebenen Richtung unterschieden. Wenn es gar keine Beschränkung in der Plattenebene gibt, so wird der Steifigkeitswert nach der Beziehung (28) eingesetzt. Wenn es eine feste Beschränkung in der Plattenebene gibt, so wird der Steifigkeitswert nach der Beziehung (24) eingesetzt.

1. INTRODUCTION

The rapid development of expressways and urban highway interchanges highlights the problems concerning the analysis of skew plate bridges. A concept in the analysis of structurally skew-anisotropic plates, taking into consideration the effect of bending-extensional coupling, will be presented herein. In 1958, Professor Lie Kuo-Hao developed a theory of skew anisotropic plates [1] (Lie's paper, written in Chinese, was summarized in detail in Ref. [2]. The expression of the deformations and the relation between the twist angle and the flexural angle, as indicated in formulae (4) and (11) of this paper were taken from Lie's paper and described in more detail by the authors.), and later in 1962, Professor Masao Naruoka [2] worked out a computer analysis in accordance with Lie's theory and verified it using model test results, as shown in Fig. 5, 6. Meanwhile, Masao Naruoka used zero torsional stiffness in his computation.

The growth of aero-space and aeronautic industries during the last two decades has evolved new materials to meet requirements of low density, high strength and orientating properties, and a new branch of mechanics was set up --- mechanics of composite materials, which developed the theory of laminated plates and shells and introduced a new concept --- the effect of bending-extensional coupling. This concept indicates that the boundary conditions within the plate plane have a great affect on nonsymmetric laminated plates [3], [4], [5].

Based on the Lie's theory, and in addition by using the concept of bending-extensional coupling, this paper studies integratively two types of boundary conditions generally encountered in skew bridge design:

(a) In the direction of the plate plane, the structurally anisotropic plate is not subject to any restraint;

(b) In the direction of the plate plane, the structurally anisotropic plate is subject to fixed restraint.

Two sets of different coefficients of stiffness have been derived for each type as mentioned above. Furthermore, in the case of the symmetrical cross-section, these two sets of coefficients are combined into one.

Firstly, under the condition of type (b), we calculated Masao Naruoka's model test [2], the results being shown in Fig. 5, 6. It was found that the theory in this paper corresponds well with test results.

Then, under the condition of type (a), we carried out the model test entirely, with the same dimensions as Masao Naruoka's model. Results evaluated according to type (a) are shown in Fig. 7, 8. Evidently, the theory in this paper also corresponds favourably with test results. This proves that the effect of bending-extensional coupling does exist.

A model test and example calculation for a structurally anisotropic plate with $\alpha=30^\circ$ is given to show that the results between types (a) and (b) differ from each other to a great extent, as shown in Fig. 10, 11.

The structurally orthotropic skew plate represents a special case of this paper. (see eqs. (34) (35)) Masao Naruoka, John B. Kennedy, et al once worked on the subject. Although the effect of bending-extensional coupling was never mentioned in their theses, the results [7] [8] of their calculations and experiments were also approximately consistent. This is due to the fact that the calculation results for skew plates under the structurally orthotropic conditions of type (a) and (b) are about the same. This could be explained by a model test we carried out along with a calculation of a structurally orthotropic skew plate with $\alpha=45^\circ$ (see Fig. 13, 14). Evidently, the difference is not so great. Apparently, it is merely a result of numerical coincidence (i.e. that under the condition of structurally orthotropic skew plates, types (a) and (b) would have no difference from each other) which resulted in the effect of bending-extensional coupling not being revealed for a long time. However the effect of bending-exten-

sional coupling is clearly demonstrated under the condition of structurally skew anisotropic plates as shown in Fig. 10, 11.
Thus a new concept is developed for the design of skew girder bridges: i. e. to determine the type of support first (whether to use neoprene bearings or steel hinged bearings) and then to calculate the internal stresses of the skew plates.

2. THE DEFORMATION OF PLATES

Considering the structurally anisotropic plate as a rigid structure made up of longitudinal ribs, transverse ribs and a covering top slab, as shown in Fig. 1. Putting X-axis in the direction of the longitudinal rib, Y-axis in the direction of transverse rib, plane XOY in the midplane of the top slab and Z-axis in the left hand coordinate system.

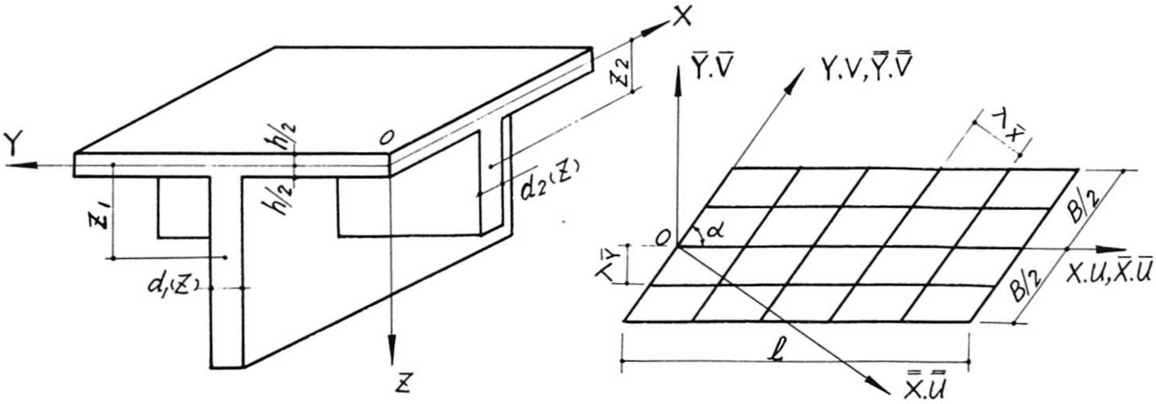

Fig. 1 Fig. 2

Denote the displacements in the X , Y directions as u , v respectively. Then, two auxiliary coordinate systems $\bar{X}o\bar{Y}$, and $\bar{\bar{X}}o\bar{\bar{Y}}$ are chosen. Make axis $\bar{X}$ coincide with axis X , axis $\bar{Y}$ perpendicular to axis X and axis $\bar{\bar{Y}}$ coincide with axis Y , axis $\bar{\bar{X}}$ perpendicular to axis $\bar{\bar{Y}}$. Let the displacements in the direction $\bar{X}$, $\bar{Y}$ equal $\bar{u}$, $\bar{v}$ and the displacements in the $\bar{\bar{X}}$, $\bar{\bar{Y}}$ direction equal $\bar{\bar{u}}$, $\bar{\bar{v}}$ respectively, as shown in Fig. 2.

let $C = \cos\alpha \qquad S = \sin\alpha$

so
$$\left.\begin{aligned} \bar{X} &= X + CY & \bar{Y} &= SY \\ \bar{\bar{X}} &= SX & \bar{\bar{Y}} &= CX + Y \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} \bar{u} &= u + Cv & \bar{v} &= Sv \\ \bar{\bar{u}} &= Su & \bar{\bar{v}} &= Cu + v \end{aligned}\right\} \tag{2}$$

In the coordinate system $\bar{X}o\bar{Y}$, the deformations may be written as

$$\begin{aligned} \bar{\varepsilon}_{\bar{x}\bar{x}} &= \partial\bar{u}/\partial\bar{x} \\ \bar{\varepsilon}_{\bar{Y}\bar{Y}} &= \partial\bar{v}/\partial\bar{Y} \\ \bar{\gamma}_{\bar{x}\bar{Y}} &= \partial\bar{v}/\partial\bar{x} + \partial\bar{u}/\partial\bar{Y} \end{aligned} \tag{3}$$

While in the coordinate system XoY, the deformations are quoted from eq. (1) of Ref. [1] i.e. in referring to Fig. 3.

$$\varepsilon_{xx} = \partial(u + c v)/\partial x$$
$$\varepsilon_{yy} = \partial(v + c u)/\partial y$$
$$\gamma_{xy} = s(\partial u/\partial y + \partial v/\partial x) \qquad (4)$$

From eqs. (1), (2), (3), (4) we obtain

$$\begin{Vmatrix} \bar{\varepsilon}_{\bar{x}\bar{x}} \\ \bar{\varepsilon}_{\bar{x}\bar{y}} \\ \bar{\gamma}_{\bar{x}\bar{y}} \end{Vmatrix} = \begin{Vmatrix} 1, & 0, & 0, \\ 0, & 1, & -c/s, \\ -c/s, & c/s, & 1, \end{Vmatrix} \begin{Vmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Vmatrix} \qquad (5)$$

This may be simplified to

$$\bar{\varepsilon} = H\ \varepsilon$$

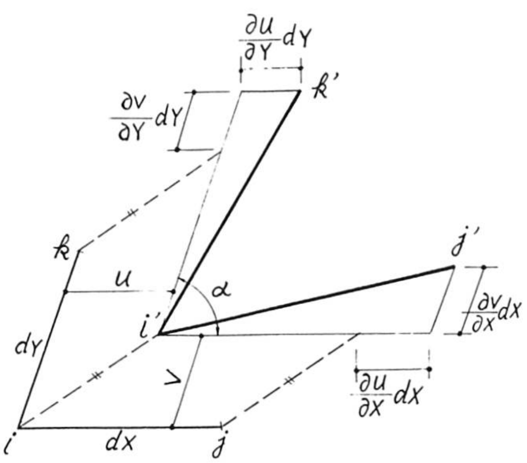

Fig. 3

Let the displacements of u , v in the plane xoy equal u_0 , v_0 .
Using the small deflection assumption, that the normal line remains straight after deflection, as follows:

$$u + c v = u_0 + c v_0 - z\ \partial w/\partial x$$
$$c u + v = c u_0 + v_0 - z\ \partial w/\partial y \qquad (6)$$

Again, using the general assumption of structurally anisotropic plates:

$$z = z_1 \qquad \varepsilon_{xx} = 0$$
$$z = z_2 \qquad \varepsilon_{yy} = 0 \qquad (7)$$

in which, z_1 = distance from the centroidal axis of main beams to plane xoy ;
z_2 = distance from the centroidal axis of cross beams to plane xoy .
see Fig. 1,
From eqs. (7), (4) yields:

$$\partial(u_0 + c v_0 - z\,\partial w/\partial x)/\partial x\Big|_{z=z_1} = 0$$

i.e.
$$u_0 + c v_0 - z_1 \partial w/\partial x = f_1(y)$$

so
$$u + c v = -(z - z_1)\,\partial w/\partial x + f_1(y)$$
$$c u + v = -(z - z_2)\,\partial w/\partial y + f_2(x) \qquad (8)$$

Substituting eq. (8) into (4) and let

$$\varphi(x, y) = (f_1'(y) + f_2'(x))/s \qquad (9)$$

so that:

$$\begin{Vmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Vmatrix} = \begin{Vmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ -c/s & -c/s, & 1/s \end{Vmatrix} \begin{Vmatrix} -(z-z_1), & 0, & 0, \\ 0 & -(z-z_2), & 0, \\ 0, & 0, & -(z-\frac{1}{2}(z_1+z_2)) \end{Vmatrix} \begin{Vmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\,\partial^2 w/\partial x \partial y \end{Vmatrix} + \begin{Vmatrix} 0, \\ 0, \\ \varphi(x,y) \end{Vmatrix} \qquad (10)$$

the above equation may be simplified to

$$\varepsilon = P A X + R$$

To obtain the twist energy later on, the relation between the twist angle and the flexural angle must be solved. Here we quote eq. (25) from Ref.[1], in other words: from the definition of twist angle, it is known that

$$\vartheta_X = \partial \bar{v}/\partial z \qquad \vartheta_Y = \partial \bar{\bar{u}}/\partial z$$

Substituting eq. (2), (6) into the above equations, we have

$$\vartheta_X = (c\, \partial w/\partial X - \partial w/\partial Y)/S$$
$$\vartheta_Y = (\partial w/\partial X - c\, \partial w/\partial Y)/S \tag{11}$$

3. RELATION BETWEEN STRESSES AND DEFORMATIONS OF PLATES

For the relation between stresses and deformations of top plates:
Denote $\bar{\sigma}$ as the stress vectors in the $\bar{X}o\bar{Y}$ coordinate system.
σ as the stress vectors in the XoY coordinate system.
Thus, in the $\bar{X}o\bar{Y}$ coordinate system, the relation between stress and strain is

$$\begin{Vmatrix} \bar{\sigma}_{\bar{x}\bar{x}}, \\ \bar{\sigma}_{\bar{Y}\bar{Y}}, \\ \bar{\tau}_{\bar{x}\bar{Y}}, \end{Vmatrix} = \frac{E}{1-\nu^2} \begin{Vmatrix} 1, & \nu, & 0, \\ \nu, & 1, & 0, \\ 0, & 0, & (1-\nu)/2, \end{Vmatrix} \cdot \begin{Vmatrix} \bar{\varepsilon}_{\bar{x}\bar{x}}, \\ \bar{\varepsilon}_{\bar{Y}\bar{Y}}, \\ \bar{\gamma}_{\bar{x}\bar{Y}}, \end{Vmatrix} \tag{12}$$

and simplified to

$$\bar{\sigma} = K\, \bar{\varepsilon}$$

According to the theory that the elastic internal energy of a structural body will not be changed irrespective of the selection of the coordinate system:

$$\frac{1}{2}\iint \bar{\varepsilon}^T \bar{\sigma}\, d\bar{x}\, d\bar{Y} = \frac{1}{2}\iint \varepsilon^T \sigma\, dx\, dY \tag{13}$$

from eq. (1) we have

$$d\bar{X}\cdot d\bar{Y} = \begin{vmatrix} 1, & C, \\ 0, & S, \end{vmatrix}\, dX \cdot dY$$

Substitute the above eq. and eqs. (5),(12) into (13) we have

$$\frac{1}{2}\iint \varepsilon^T (H^T K H S)\varepsilon\, dx dY = \frac{1}{2}\iint \varepsilon^T \sigma\, dx dY \tag{14}$$

so $$\sigma = (H^T K H \cdot S)\varepsilon = (SE/1-\nu^2)\, L \cdot \varepsilon$$

in which
$$L = \begin{Vmatrix} 1+[C^2(1-\nu)/2S^2], & \nu-[C^2(1-\nu)/2S^2], & -C(1+\nu)/2S, \\ \nu-[C^2(1-\nu)/2S^2], & 1+[C^2(1-\nu)/2S^2], & -C(1+\nu)/2S, \\ -C(1+\nu)/2S, & -C(1+\nu)/2S, & \frac{C^2}{S^2}+\frac{(1-\nu)}{2}, \end{Vmatrix} \tag{15}$$

For the relation between stresses and strains of longitudinal ribs and transverse ribs:
In the Z direction, take out a thin sheet of longitudinal and transverse ribs with the depth dZ . Let $d_1(Z)$ be expressed as the width of the longitudinal rib at Z and $d_2(Z)$ as the width of the transverse rib at Z , Fig. 1. by using the relation between stresses and strains and recognizing that both longitudinal and transverse ribs are all very thin in their transverse direction respectively, the affects of γ_{XY} against τ_{XY} may be neglected. Thus

$$\left\| \begin{matrix} \sigma_{XX}, \\ \sigma_{YY}, \\ \tau_{XY}, \end{matrix} \right\| = \frac{SE}{1-\nu^2} \left\| \begin{matrix} d_1(Z)/\lambda_{\bar{Y}}, & 0, & 0, \\ 0 & , d_2(Z)/\lambda_{\bar{X}}, & 0, \\ 0 & , 0 & , 0, \end{matrix} \right\| \cdot \left\| \begin{matrix} \varepsilon_{XX}, \\ \varepsilon_{YY}, \\ \gamma_{XY}, \end{matrix} \right\| \quad (16)$$

For relation between twist moment and twist angle of longitudinal and transverse ribs:
Let the twist moment of longitudinal rib be as T_X , the twist rigidity as K_X , the twist moment of transverse rib be as T_Y and the twist rigidity as K_Y , thus

$$\left\| \begin{matrix} T_X/\lambda_{\bar{Y}}, \\ T_Y/\lambda_{\bar{X}}, \end{matrix} \right\| = \left\| \begin{matrix} K_X/\lambda_{\bar{Y}} & , 0 & , \\ 0 & , K_Y/\lambda_{\bar{X}} \end{matrix} \right\| \cdot \left\| \begin{matrix} \partial \vartheta_X/\partial X, \\ \partial \vartheta_Y/\partial Y, \end{matrix} \right\| \quad (17)$$

4. EXPRESSIONS OF POTENTIAL ENERGY

For the bending energy of top plate:

$$U_1 = \frac{1}{2}\iiint \varepsilon^T \sigma \, dX dY dZ = \frac{1}{2}\iiint (R+PA\mathcal{X})^T L (R+PA\mathcal{X})(SE/1-\nu^2)\, dX dY dZ$$
$$= \frac{1}{2}\iiint (R^T L R + 2 R^T L P A \mathcal{X} + \mathcal{X}^T A^T P^T L P A \mathcal{X})(SE/1-\nu^2) dX dY dZ \quad (18)$$

Then we can easily obtain $\int R^T L R \, dZ = \varphi^2(X,Y)\left[C^2/S^2 + (1-\nu)/2\right] h$

$$\int R^T L P A \mathcal{X} \, dZ = \varphi(X,Y)\, h\, G\, \mathcal{X}$$

in which $G = \left\| -C Z_1/S^3, \; -C Z_2/S^3, \; \left[C^2/S^2 + (1-\nu)/2\right](Z_1+Z_2)/2S, \right\|$ (19)

$$\int \mathcal{X}^T A^T P^T L P A \mathcal{X} dZ = \frac{\mathcal{X}^T}{S^4} \left\| \begin{matrix} (\frac{h^3}{12} + Z_1^2 h), & & \text{symmetric} \\ (C^2+S^2\nu)(\frac{h^3}{12} + Z_1 Z_2 h), & (\frac{h^3}{12} + Z_2^2 h) & \\ -C(\frac{h^3}{12} + Z_1 \frac{(Z_1+Z_2)}{2} h), & -C(\frac{h^3}{12} + Z_2 \frac{(Z_1+Z_2)}{2} h), & \frac{1+C^2-S^2\nu}{2}(\frac{h^3}{12} + (\frac{Z_1+Z_2}{2})^2 h), \end{matrix} \right\| \mathcal{X} \quad (20)$$

For the bending energy in the longitudinal ribs and transverse ribs:
From eqs. (10), (16) we have

$$U_2 = \frac{1}{2}\iiint \varepsilon^T \sigma \, dX dY dZ = \frac{SE}{2(1-\nu^2)} \iiint (PA\mathcal{X}+R)^T \left\| \begin{matrix} d_1(Z)/\lambda_{\bar{Y}}, & 0 & , 0, \\ 0 & , d_2(Z)/\lambda_{\bar{X}}, & 0, \\ 0 & , 0 & , 0, \end{matrix} \right\| (PA\mathcal{X}+R) dX dY dZ$$

note that
$$\int d_1(z)(z-z_1)^2 dz = I_x/\lambda_{\bar{y}} - h^3/12 - h z_1^2$$
$$\int d_2(z)(z-z_2)^2 dz = I_Y/\lambda_{\bar{x}} - h^3/12 - h z_2^2$$
in which I_x, I_Y = moment of inertias of main and cross beams respectively

$$U_2 = \frac{1}{2}\iint \chi^T \begin{Vmatrix} \left(\frac{S^4 I_x}{\lambda_{\bar{y}}} - \frac{S^4 h^3}{12} - S^4 h z_1^2\right), & 0, & 0, \\ 0, & \left(\frac{S^4 I_Y}{\lambda_{\bar{x}}} - \frac{S^4 h^3}{12} - S^4 h z_2^2\right), & 0, \\ 0, & 0, & 0, \end{Vmatrix} \chi \frac{S E dx dY}{S^4(1-\nu^2)} \tag{21}$$

For the twist energy of longitudinal ribs and transverse ribs:
From eq. (11) we have

$$U_3 = \frac{1}{2}\iint \left\| \partial\vartheta_x/\partial x, \partial\vartheta_Y/\partial Y \right\| \cdot \begin{Vmatrix} K_x/\lambda_{\bar{y}}, & 0, \\ 0, & K_Y/\lambda_{\bar{x}}, \end{Vmatrix} \cdot \begin{Vmatrix} \partial\vartheta_x/\partial x, \\ \partial\vartheta_Y/\partial Y, \end{Vmatrix} S\, dx dY$$

$$= \frac{1}{2S^4}\iint \chi^T \begin{Vmatrix} C^2 S^2 K_x/\lambda_{\bar{y}}, & 0, & C S^2 K_x/2\lambda_{\bar{y}}, \\ 0, & C^2 S^2 K_Y/\lambda_{\bar{x}}, & C S^2 K_Y/2\lambda_{\bar{x}}, \\ C S^2 K_x/2\lambda_{\bar{y}}, & C S^2 K_Y/2\lambda_{\bar{x}}, & \frac{S^2}{4}\left(\frac{K_x}{\lambda_{\bar{y}}} + \frac{K_Y}{\lambda_{\bar{x}}}\right), \end{Vmatrix} \chi\, S\, dx dY \tag{22}$$

For total potential energy:

$$U = U_1 + U_2 + U_3 - \iint q\, \omega\, S\, dx dY = \frac{1}{2S^4}\iint \{\chi^T D_1 \chi + [2S^4 E h\, \varphi(x,Y)/1-\nu^2] G\chi$$
$$+ [S^4 E h\, \varphi^2(x,Y)/1-\nu^2](C^2/S^2 + (1-\nu)/2)\}\, S\, dx dY - \iint q\, \omega\, S\, dx dY \tag{23}$$

in which

$$D_1 = \begin{Vmatrix} D'_{11}, & D'_{12}, & D'_{13}, \\ D'_{21}, & D'_{22}, & D'_{23}, \\ D'_{31}, & D'_{32}, & D'_{33}, \end{Vmatrix}$$

$$D'_{11} = [S^4 E I_x/\lambda_{\bar{y}}(1-\nu^2)] + C^2[(1+S^2)[(E z_1^2 h/1-\nu^2) + \bar{D}] + (S^2 K_x/\lambda_{\bar{y}})]$$

$$D'_{12} = D'_{21} = (C^2 + S^2\nu)[(E z_1 z_2 h/1-\nu^2) + \bar{D}]$$

$$D'_{22} = [S^4 E I_Y/\lambda_{\bar{x}}(1-\nu^2)] + C^2[(1+S^2)[(E z_2^2 h/1-\nu^2) + \bar{D}] + (S^2 K_Y/\lambda_{\bar{x}})]$$

$$D'_{13} = D'_{31} = -C[(E z_1 (z_1+z_2)h/2(1-\nu^2)) + \bar{D} + (S^2 K_x/2\lambda_{\bar{y}})]$$

$$D'_{23} = D'_{32} = -C[(E z_2 (z_1+z_2)h/2(1-\nu^2)) + \bar{D} + (S^2 K_Y/2\lambda_{\bar{x}})]$$

$$D'_{33} = \frac{1+C^2-S^2\nu}{2}\left[\frac{E}{1-\nu^2}\left(\frac{z_1+z_2}{2}\right)^2 h + \bar{D}\right] + \frac{S^2}{4}\left(\frac{K_x}{\lambda_{\bar{y}}} + \frac{K_Y}{\lambda_{\bar{x}}}\right)$$

$$\bar{D} = E h^3/12(1-\nu^2) \tag{24}$$

5. THE EFFECT OF BENDING-EXTENSIONAL COUPLING

Obviously, from eq. (23), the potential energy U contains as arbitrary function $\varphi(X,Y)$, which makes ω indefinite and will be definitely determined only after $\varphi(x,Y)$ has been defined according to the condition requireed in the plate plane. It is mainly caused by the non-symmetry of the sections, which we have called the effect of bending-extensional coupling.
Now let's study the two types of conditions often encountered in practical design:
(a) In the direction of the plate plane, the structurally anisotropic plate is not subject to any restraint.
Since the plate is not subject to any restraint in the direction of the plate plane, it can be considered that the midplane of the top plate will present a rigid body movement after it is loaded.
Therefore, we have

$$\gamma_{XY}\Big|_{Z=0} = 0 . \tag{25}$$

From eq. (10),

$$\gamma_{XY}\Big|_{Z=0} = \left\| -CZ_1/S\ ,\ -CZ_2/S\ ,\ (Z_1+Z_2)/2S\ . \right\| \mathcal{X} + \varphi(X,Y) = 0$$

so
$$\varphi(X,Y) = \left\| CZ_1/S\ ,\ CZ_2/S\ ,\ -(Z_1+Z_2)/2S\ . \right\| \mathcal{X} \tag{26}$$

substituting in eq. (23), we have

$$U = \frac{1}{2S^4}\iint \mathcal{X}^T D_0 \mathcal{X}\, S\, dx\, dY - \iint q\, \omega\, S\, dx\, dY \tag{27}$$

in which, the matrix D_0 is defined as

$$D^{\circ}_{11} = (S^4 E I_X/\lambda_{\bar{Y}}(1-\nu^2)) + (1-S^4)\bar{D} + C^2S^2[(EZ_1^2h/2(1+\nu)) + (K_X/\lambda_{\bar{Y}})]$$
$$D^{\circ}_{12} = D^{\circ}_{21} = (C^2+S^2\nu)\bar{D} + [S^2(\nu-S^2\nu-C^2)EZ_1Z_2h/2(1-\nu^2)]$$
$$D^{\circ}_{22} = (S^4 E I_Y/\lambda_{\bar{X}}(1-\nu^2)) + (1-S^4)\bar{D} + C^2S^2[(EZ_2^2h/2(1+\nu)) + (K_Y/\lambda_{\bar{X}})]$$
$$D^{\circ}_{13} = D^{\circ}_{31} = -C(\bar{D} + (S^2K_X/2\lambda_{\bar{Y}}))$$
$$D^{\circ}_{23} = D^{\circ}_{32} = -C(\bar{D} + (S^2K_Y/2\lambda_{\bar{X}}))$$
$$D^{\circ}_{33} = (\bar{D}(1+C^2-S^2\nu)/2) + \{S^2[(K_X/\lambda_{\bar{Y}}) + (K_Y/\lambda_{\bar{X}})]/4\} \tag{28}$$

We can obtain the solution of non-restrained ω by taking the variation of $U=0$, i.e. $\delta U=0$. This solution is equivalent to a 4th order partial differential equation together with its corresponding boundary conditions, which is exactly the same as in Ref.[1], [2], except that D_0 was expressed by eq. (28). This is very similar to the "reduced bending stiffness" of laminated plates of composite materials.
(b) In the direction of the plate plane, the structurally anisotropic plate is subject to fixed restraint.
i.e.

$$X=0\ ,\quad X=l\ ,\qquad Z=Z_1 \qquad u=0\ ,\quad V=0.$$
$$Y=-\frac{B}{2}\ ,\quad Y=+\frac{B}{2}\ ,\qquad Z=Z_2 \qquad u=0\ ,\quad V=0.$$

substituting in eq. (8), we have

$$f_1(Y)=0\ ,\qquad f_2(X)=0 .$$

$$\varphi(x,y) = (f_1'(y) + f_2'(x)) \cdot \frac{1}{S} = 0$$

substituting $\varphi(x,y)$ into eq. (23), we have

$$U = \frac{1}{2S^4} \iint \chi^T D_1 \chi \, S \, dx\,dy - \iint q\,\omega\, S \, dx\,dy \tag{29}$$

Taking $\delta U = 0$, we can thus obtain the solution of ω in the fixed condition. This solution is also equivalent to a 4th order partial differential equation together with its corresponding boundary conditions which is exactly the same as in Ref. [1], [2], except that the coefficient is expressed as in eq. (24) instead.

6. PARTICULAR CONDITIONS

As in Ref. [1], we use orthotropic plates, isotropic plates and beam grillages for verification.

For the orthotropic plate:
If it is not restrained in the direction of the plate plane:
In eq. (28) let $\alpha = 90^\circ$, $C = 0$, $S = 1$

$$D_0 = \begin{Vmatrix} EI_x/\lambda_{\bar{y}}(1-\nu^2) , & \nu[\bar{D} + (Ehz_1z_2/(1-\nu^2))] , & 0 , \\ \nu[\bar{D} + (Ehz_1z_2/(1-\nu^2))] , & EI_y/\lambda_{\bar{x}}(1-\nu^2) , & 0 , \\ 0 , & 0 , & \frac{1-\nu}{2}\bar{D} + \frac{1}{4}\left(\frac{K_x}{\lambda_{\bar{y}}} + \frac{K_y}{\lambda_{\bar{x}}}\right), \end{Vmatrix} \tag{30}$$

If it is fixed in the direction of the plate plane:
In eq. (24) let

$$D_1 = \begin{Vmatrix} EI_x/\lambda_{\bar{y}}(1-\nu^2) , & \nu[\bar{D} + (Ehz_1z_2/(1-\nu^2))] , & 0 , \\ \nu[\bar{D} + (Ehz_1z_2/(1-\nu^2))] , & EI_y/\lambda_{\bar{x}}(1-\nu^2) , & 0 , \\ 0 , & 0 , & \frac{1-\nu}{2}\left[\bar{D} + \frac{Eh}{1-\nu^2}\left(\frac{z_1+z_2}{2}\right)^2 + \frac{1}{4}\left(\frac{K_x}{\lambda_{\bar{y}}} + \frac{K_y}{\lambda_{\bar{x}}}\right)\right] \end{Vmatrix} \tag{31}$$

For skew isotropic plate:

In eqs. (24), (28) let $z_1 = z_2 = 0$

Then the two eqs. are equivalent and equal to

$$D_1 = D_0 = \begin{Vmatrix} \bar{D} , & (C^2+S^2\nu)\bar{D} , & -C\bar{D} , \\ (C^2+S^2\nu)\bar{D} , & \bar{D} , & -C\bar{D} , \\ -C\bar{D} , & -C\bar{D} , & (1+C^2-S^2\nu)\bar{D}/2 . \end{Vmatrix} \tag{32}$$

It can be seen from eq. (32) that the effect of bending-extensional coupling disappeared at this stage.

For skew plane beam grillage:

In eqs. (24), (28) let $h = 0$, $z_1 = z_2 = 0$

Then the two eqs. are equivalent, both being equal to

$$D_1 = D_0 = \begin{Vmatrix} \frac{S^4 E I_x}{\lambda_{\bar{Y}}(1-\nu^2)} + \frac{C^2 S^2 K_x}{\lambda_{\bar{Y}}}, & 0, & -\frac{C S^2 K_x}{2\lambda_{\bar{Y}}}, \\ 0, & \frac{S^4 E I_Y}{\lambda_{\bar{x}}(1-\nu^2)} + \frac{C^2 S^2 K_Y}{\lambda_{\bar{x}}}, & -\frac{C S^2 K_Y}{2\lambda_{\bar{x}}}, \\ -\frac{C S^2 K_x}{2\lambda_{\bar{Y}}}, & -\frac{C S^2 K_Y}{2\lambda_{\bar{x}}}, & \frac{S^2}{4}\left(\frac{K_x}{\lambda_{\bar{Y}}} + \frac{K_Y}{\lambda_{\bar{x}}}\right), \end{Vmatrix} \tag{33}$$

It can be seen from eq. (33) that the effect of bending-extensional coupling disappeared at this stage.

For the orthotropic skew plate:
From eq. (1) we obtain

$$\begin{Vmatrix} \frac{\partial^2 w}{\partial \bar{x}^2}, \\ \frac{\partial^2 w}{\partial \bar{Y}^2}, \\ 2\frac{\partial^2 w}{\partial \bar{x}\partial \bar{Y}}, \end{Vmatrix} = \begin{Vmatrix} 1, & 0, & 0, \\ \frac{C^2}{S^2}, & \frac{1}{S^2}, & -\frac{C}{S^2}, \\ -\frac{2C}{S}, & 0, & \frac{1}{S}. \end{Vmatrix} \cdot \begin{Vmatrix} \frac{\partial^2 w}{\partial X^2}, \\ \frac{\partial^2 w}{\partial Y^2}, \\ 2\frac{\partial^2 w}{\partial X \partial Y}. \end{Vmatrix}$$

simplified to: $\bar{X} = J X$

By using the theorem of energy we can find that,
under the condition where there is no restraint in the direction of the plate plane, the rigidity of orthotropic skew plate is:

$$D = J^T D_0 J \qquad \text{(here } D_0 \text{ is expressed by eq. (30))} \tag{34}$$

under the condition where the plate is fixed in the direction of the plate plane, the rigidity of orthotropic skew plate is:

$$D = J^T D_1 J \qquad \text{(here } D_1 \text{ is expressed by eq. (31))} \tag{35}$$

7. EVALUATION

Various methods of evaluation, such as the method of finite elements [6] , the method of series [8] and the method of difference [2] etc. have been developed by several authors. In coping with the requirements of design, the method of variation is to be used in this paper for the conditions of two opposite simply supported edges and two opposite free edges. This solution has been already programmed. As shown in Fig. 2, the stiffened beam with bending rigidity EI is supported in the plane where $Y=\pm B/2$. If a concentrated load acts at the point (x_0, Y_0), then q can be expressed by the Dirac function.
So eqs. (23), (27) can be rewritten as

$$U = \frac{1}{2S^4}\iint X^T D X \, S \, dx\, dY + \int_{Y=\pm B/2} \frac{EI}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx - \int P_0\, \delta(x-x_0)\,\delta(Y-Y_0)\cdot\frac{1}{S}\cdot S\, w\, dx\, dY \tag{36}$$

let $$w = \| Y^0, Y^1, \cdots\cdots Y^n \| \cdot \| f_0(x), f_1(x) \cdots\cdots f_n(x) \|^T = Y_n^T \cdot f \tag{37}$$

substitute eq. (37) into (36), yields

$$U=\frac{1}{2S}\int_0^{l}\left\{\begin{Vmatrix} f''\\ f\\ 2f'\end{Vmatrix}^{T}\left[\int\begin{Vmatrix} D_{11}Y_n^TY_n, & D_{12}Y_n^TY_n'', & D_{13}Y_n^TY_n'\\ D_{21}Y_n''^TY_n, & D_{22}Y_n''^TY_n'', & D_{23}Y_n''^TY_n'\\ D_{31}Y_n'^TY_n, & D_{32}Y_n'^TY_n'', & D_{33}Y_n'^TY_n'\end{Vmatrix} S\,dY\right]\begin{Vmatrix} f''\\ f\\ 2f'\end{Vmatrix}\right.$$

$$\left.+EI\,(Y_n^Tf'')^2S^4\Big|_{Y=\pm\frac{B}{2}}-2P_0\,\delta(x-x_0)\,Y_n^Tf\,S^4\Big|_{Y=Y_0}\right\}dx=\int F(x,f,f',f'')\,dx$$

$$\delta U=0$$

$$\delta U=\int_0^{l}\left(\frac{\partial F}{\partial f}-\frac{d}{dx}\frac{\partial F}{\partial f'}+\frac{d^2}{dx^2}\frac{\partial F}{\partial f''}\right)\delta f\,dx+\left\{\frac{\partial F}{\partial f'}-\frac{d}{dx}\frac{\partial F}{\partial f''}\right\}\delta f\Big|_0^{l}+\frac{\partial F}{\partial f''}\,\delta f'\Big|_0^{l}=0. \quad (38)$$

Thus the solution can be obtained by simplifying the 4th order partial differential equation to a set of ordinary differential equations. Due to limited space, details of solution and programms are omitted.

8. COMPARISON BETWEEN MODEL TESTS AND CALCULATED RESULTS

8.1. Masao Naruoka's model test

Dimensions of model are shown in Fig. 4. There are stiffened edge beams with bending rigidity

$$EI\Big|_{Y=\pm\frac{B}{2}}=0.234\ E.$$

From eq. (24) yields:

$$D_1=\begin{Vmatrix} 0.10010, & 0.01422, & 0.02523,\\ 0.01422, & 0.02381, & 0.01066,\\ 0.02523, & 0.01066, & 0.01668,\end{Vmatrix}E$$

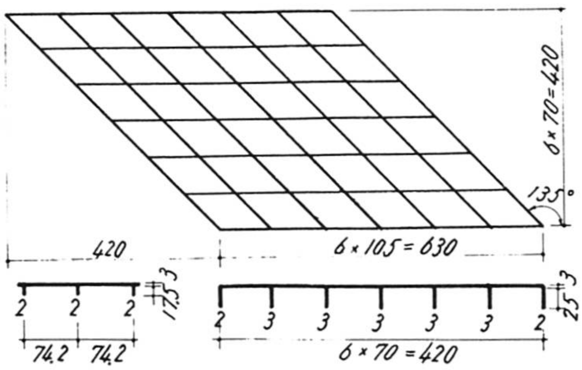

Fig. 4

Concentrated load P_0 acted separately on $X_0=0$; $Y_0=0$, $B/6$, $B/3$, $B/2$. The results of computation of midspan deflections ω, internal forces M, and the comparative results between experimental and calculated values mentioned in Ref. [2] are shown in Fig. 5, 6 respectively.

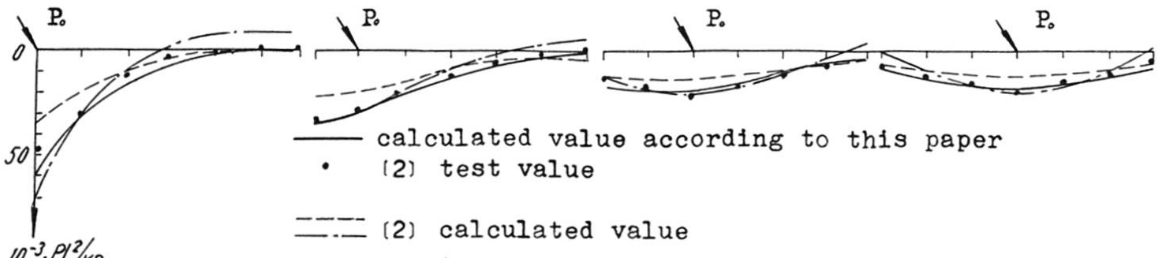

Fig. 5 Midspan deflection

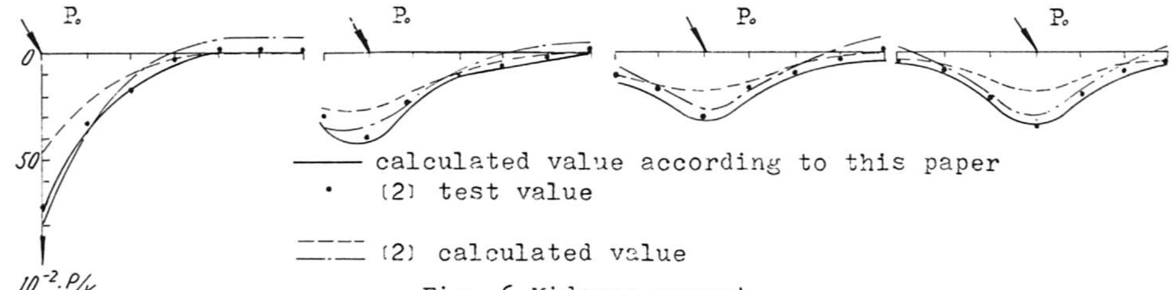

Fig. 6 Midspan moment

Another experiment was performed entirely in accordance with the above mentioned dimensions of Masao Naruoka's model, but it was not subject to restraint in the direction of the plate plane.
This is equivalent to the type (a) of the said ehtory. Eq. (28) yields

$$D_o = \begin{Vmatrix} 0.06811, & 0.00213, & 0.00205, \\ 0.00213, & 0.01924, & 0.00190, \\ 0.00205, & 0.00190, & 0.00188, \end{Vmatrix} E$$

For the comparative results between the calculated and experimental values of deflections ω and internal forces M , see Fig. 7, 8.

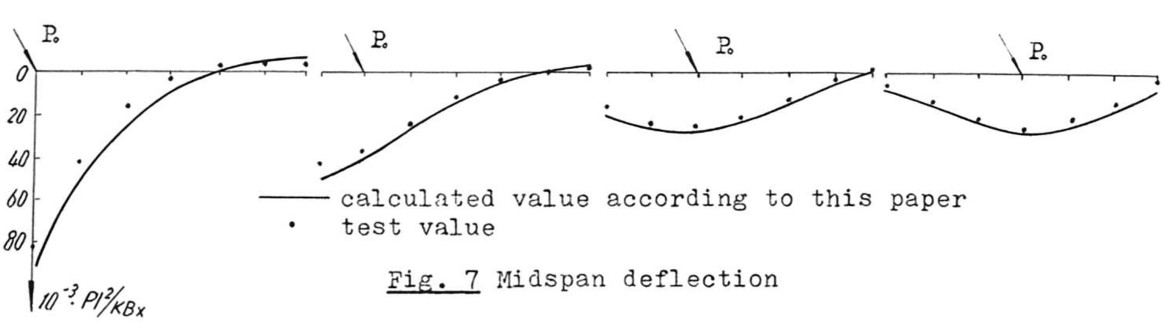

Fig. 7 Midspan deflection

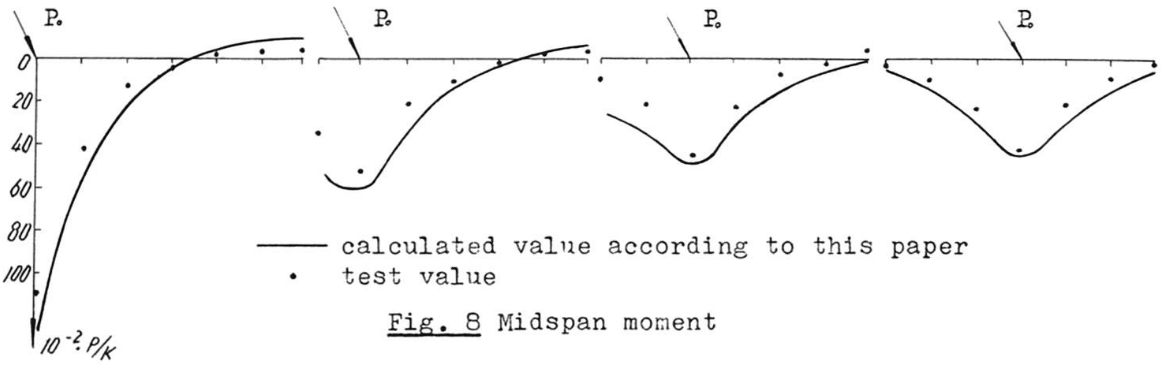

Fig. 8 Midspan moment

From the above two examples, we can see that different results will be yielded from different boundary conditions in the direction of the plate plane in the same Masao Naruoka's model, and both theory and practice are consistant.

8.2. Model test with $\alpha = 30°$

The material used is plexiglass with $E = 2.45 \times 10^4$ Kg/cm^2, $\nu = 0.38$. Dimensions of the model are shown in Fig. 9.

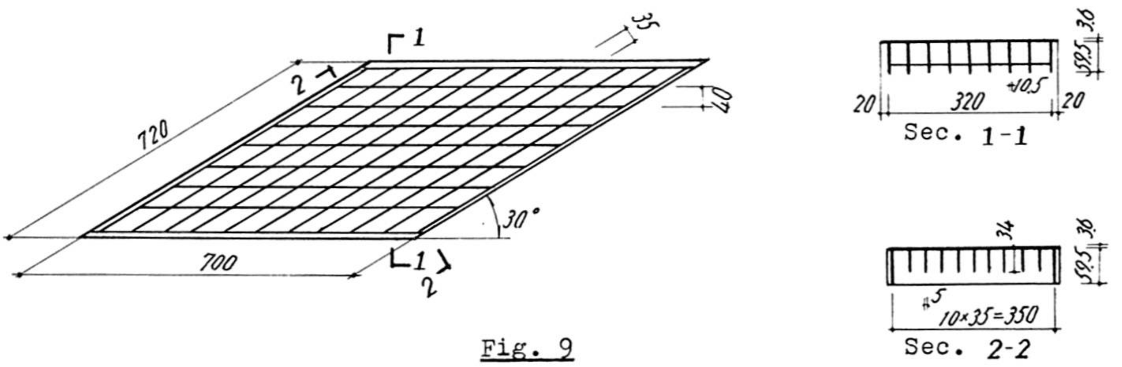

Fig. 9

The experiment was carried out as type (a), hence the calculation was worked out accordingly, i.e. to calculate by eq. (28).

$$D_o = \begin{Vmatrix} +0.74928, & -0.03620, & -0.02393, \\ -0.03620, & +0.12311, & -0.00537, \\ -0.02393, & -0.00537, & +0.01613, \end{Vmatrix} E$$

In comparing the calculated and experimental values of the midspan deflections ω and internal forces M as shown in Fig. 10, 11, a favourable consistence is found.

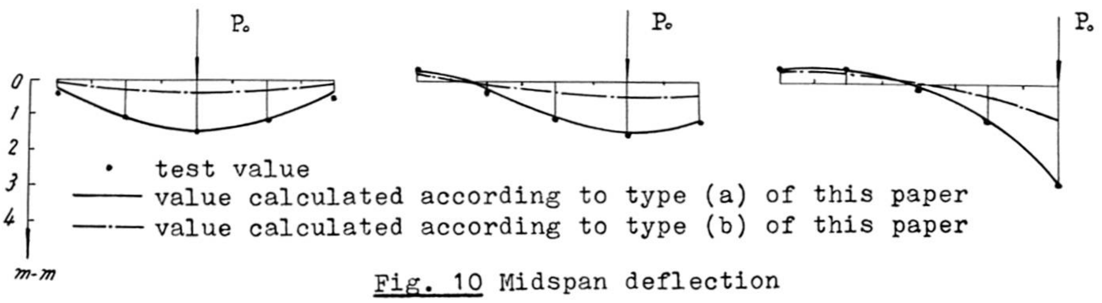

Fig. 10 Midspan deflection

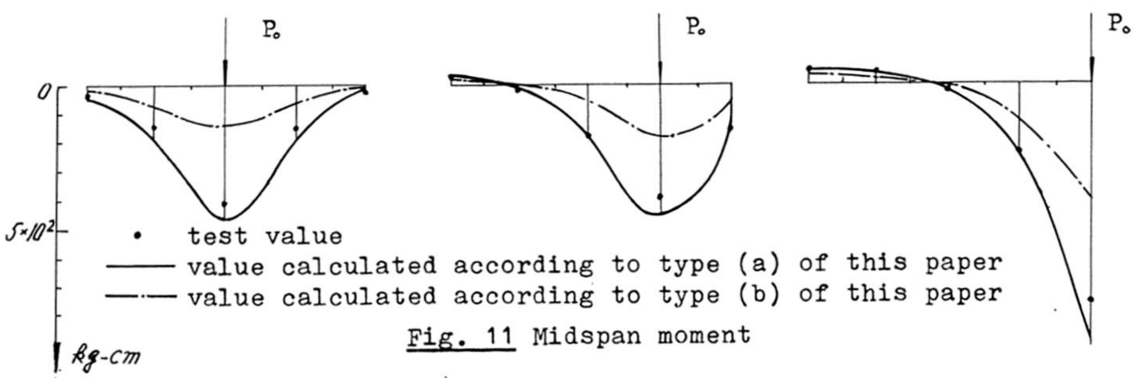

Fig. 11 Midspan moment

If we don't distinguish type (a) from type (b) and misuse eq. (24) (i.e. type (b)) to calculate the rigidity, where

$$D_1 = \begin{Vmatrix} +3.18174, & +0.98813, & -1.72609, \\ +0.98813, & +0.55446, & -0.72217, \\ -1.72609, & -0.72217, & +1.17186, \end{Vmatrix} E$$

then, the calculated values of the midspan deflections ω and internal forces M, represented by dotted lines in Fig. 10, 11 show up a tremendous difference between the results of types (a) and (b).

8.3. Experiment of a structurally orthotropic skew plate with $\alpha = 45°$

The material used is plexiglass with $E=2.45\times10^4 Kg/cm^2$ $\nu=0.38$
Dimensions of model are shown in Fig. 12.

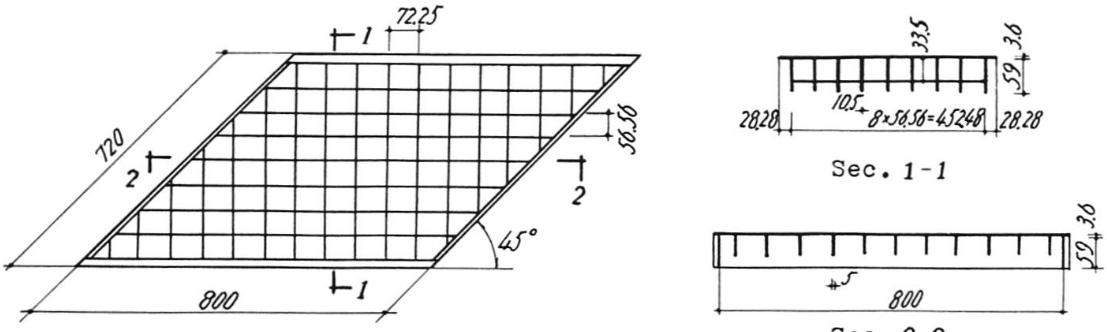

Fig. 12

The experiment was carried out under the condition of type (a), hence the calculation was worked out accordingly, i.e. using eq. (34) for computation:

$$D_o = \left\| \begin{matrix} +2.08274, & +0.54762, & -0.41215, \\ +0.54762, & +0.82098, & -0.58052, \\ -0.41215, & -0.58052, & +0.42811, \end{matrix} \right\| E$$

Comparison between the calculated and experimental values of midspan deflections ω and internal forces M, as shown in Fig. 13, 14, is obviously quite in consistence.

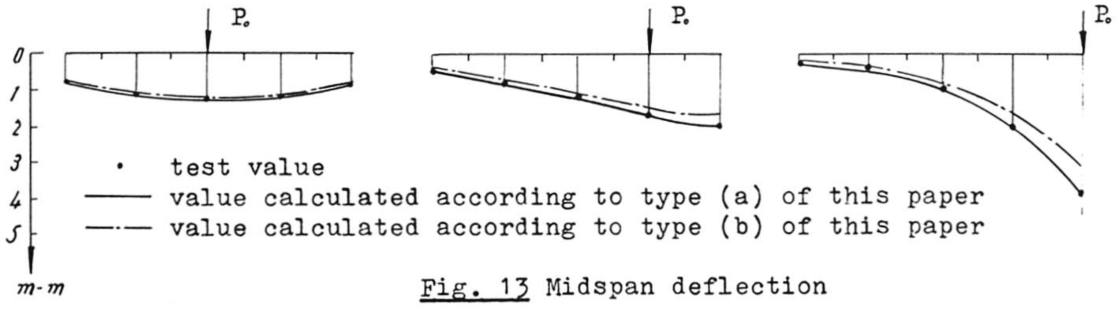

Fig. 13 Midspan deflection

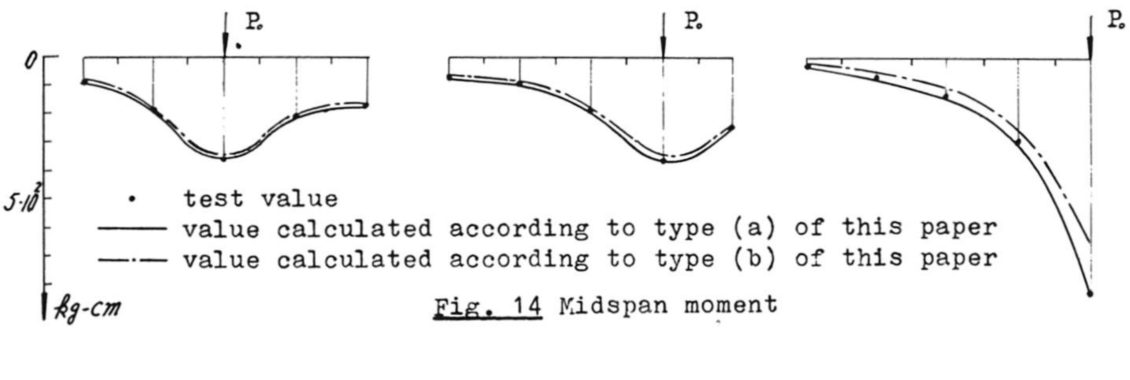

Fig. 14 Midspan moment

When the calculation was carried out under type (b), i.e. using eq. (35):

$$D_1 = \begin{Vmatrix} +2.39194, & +0.54762, & -0.63079, \\ +0.54762, & +0.82098, & -0.58052, \\ -0.63079, & -0.58052, & +0.58271, \end{Vmatrix} E$$

then the calculated values of midspan deflections ω and internal forces M were shown by dotted lines in Fig. 13, 14. It is obvious that although the conditions are entirely different under types (a) and (b) for the structurally orthotropic skew plates, the differences in the calculated results are insignificant. In short, it is clear from the above four examples that this theory is relatively consistant with experimental results.

Besides the above mentioned tests, the authors have made a series of other model tests (including $\alpha = 30°$, $45°$, $60°$, $90°$). Due to limited space, they will not be enumerated here one by one. Generally speaking, except for specific points, the difference in deflection is less than 8%, and the difference in moment is less than 15%.

9. CONCLUSION

For structurally skew anisotropic plate, it is important to distinguish the condition of restraint in the direction of the plate plane.
If it is not subject to any restraint in the direction of the plate plane (in the case, where a neoprene bearing with comparatively small shear modulus is used), the stiffness coefficients of eq. (28) or (34) (orthotropic skew plate) should be used for computation. If it is subject to fixed restraint in the direction of the plate plane, the stiffness coefficients of eq. (24) or (35) (orthotropic skew plate) should be used for computation.

NOTATIONS

α.	skew angle
C, S.	$C = \cos\alpha$ $S = \sin\alpha$
$X, Y, \bar{X}, \bar{Y}, \bar{\bar{X}}, \bar{\bar{Y}}, Z$.	coordinates
$u, v, \bar{u}, \bar{v}, \bar{\bar{u}}, \bar{\bar{v}}, \omega$.	displacements
Z_1, Z_2	distance from the centroidal axis of main and cross beams to plane
ℓ. B.	the length and width in the X and Y direction of the plate
h.	thickness of the top plate
$d_1(Z)$, $d_2(Z)$.	the width of longitudinal and transverse beams at Z
$\lambda_{\bar{X}}$, $\lambda_{\bar{Y}}$.	the distance between longitudinal beams and distance between cross beams
$\varepsilon_{XX}, \varepsilon_{YY}, \gamma_{XY}, \bar{\varepsilon}_{\bar{X}\bar{X}}, \bar{\varepsilon}_{\bar{Y}\bar{Y}}, \bar{\gamma}_{\bar{X}\bar{Y}}$.	strain
$\sigma_{XX}, \sigma_{YY}, \tau_{XY}, \bar{\sigma}_{\bar{X}\bar{X}}, \bar{\sigma}_{\bar{Y}\bar{Y}}, \bar{\tau}_{\bar{X}\bar{Y}}$.	stress
ϑ_X, ϑ_Y.	twist angle
T_X, T_Y.	twist moment of longitudinal rib and transverse rib
K_X, K_Y	twist rigidity of longitudinal rib and transverse rib
I_X, I_Y	moment of inertia of longitudinal beam and transverse beam

I.	moment of inertia of stiffened side beam
E.	modulus of elasticity
ν.	Poisson's ratio
$\bar{D} = \frac{Eh^3}{12(1-\nu^2)}$.	bending rigidity of top plate
U, U_1, U_2, U_3.	potential energy, internal energy
P_0, q.	external loading
X_0, Y_0.	point of action of concentrated force
D_0.	rigidity matrix in the direction of the plate plane not subject to any restraint
D_1.	rigidity matrix in the direction of the plate plane subject to restraint
$\sigma, \bar{\sigma}$.	stress vector
J.	curvature transformation matrix (from XOY coordinate system to $\bar{X}O\bar{Y}$ coordinate system)
$\bar{\varepsilon}, H, \varepsilon$.	see eq. (5)
P, A, X, R.	see eq. (10)
K.	see eq. (12)
L.	see eq. (15)
G.	see eq. (19)

ACKNOWLEDGEMENT

The authors are greatly indebted to Prof. Lie Kuo-Hao of Tong Ji University for his valuable suggestions made both in basic concepts and tests.

The computer programme of this paper was worked out by Engr. Mo Shan-Jian in collaboration with the Shanghai Computer Technique Research Institute.

REFERENCES

1. Lie Kuo-Hao: "Theorie der schiefwinklig-anisotropen Platte und ihre Anwendung auf schiefe Brücken" Acta Mechanica Sinica 2.1 1958, P.78-88.
2. Masao Naruoka: "Über die Berechnung schiefer anisotroper Platten" Der Bauingenieur 1962, 11.
3. C.M. Bert, P.H. Francis: "Composite Material Mechanics: Structural Mechanics" A.I.A.A. Vol. 12, No.9. 1974.
4. J.M. Whitney: "Bending-Extensional Coupling in Laminated Plates under Transverse Loading" Journal Of Composite Materials Vol. 3, 1969.
5. J.E. Ashton, J.M. Whitney: "Theory of Laminated Plates" Stanford Conn. Technomic. 1970.
6. O.C. Zienkiewicz, Y.K. Cheung: "The Finite Element Method in Structural and Continuum Mechanics" McGraw Hill. 1967.
7. Masao Naruoka, Hiroshi Ohmura: "On the Analysis of A Skew Girder Bridge by the Theory of Orthotropic Parallelogram Plates" I.A.B.S.E. 1959-1960.
8. John B. Kennedy: "Bending of Skew Orthotropic Plate Structures" Journal of the Structural Division A.S.C.E. 1976.8.
9. S. Timoshenko, S. Woinowsky-Kriger: "Theory of Plates and Shells" McGraw Hill, 2nd. ed. 1959.