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A Study of Cyclic Plasticity

Une étude sur la plasticité en régime cyclique

Eine Studie über zyklische Plastizität

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1. INTRODUCTION

The plastic behaviour of structures under monotonic loading is usually predicted with sufficient accuracy by simple theories of plasticity such as flow rules associated with a single yield surface. For varying loads, however, these theories do not represent the complex plastic behaviour with sufficient fidelity. The theory of plasticity must be modified by introducing a suitable set of internal state parameters that enter into the yield condition and the constitutive relations. The mathematical structure can become quite complicated, as has been demonstrated [1], and there is a particular need for simpler material models that reflect the most essential aspects of plastic behaviour for a reasonably wide class of problems but have sufficient simplicity and accuracy for practical design. It is the aim of this paper to discuss a simplified theory of cyclic plasticity suitable for solving boundary value problems of small deformation behaviour of structures under cyclic proportional loading. We present a stress-strain relationship suitable for cyclic proportional stressing employing the Masing [2] hardening rule and two scalar state parameters (equivalent stress, or equivalent strain, at the last two reversals) and show that a wide class of problems of cyclic loading can be solved for this representation. For line or surface structures the formulation is readily transformed in terms of generalized stresses and strains.

2. STRESS STRAIN RELATIONS

Consider a material with the uniaxial hardening curve

$$\sigma = f(\epsilon) \tag{1}$$

relating stress σ and strain ϵ by a monotonic odd function f, with a reverse loading curve given by the "Masing transformation" of Eq. 1, viz.

$$\left.\begin{aligned} \sigma - \sigma^- &= 2f(\tfrac{1}{2}[\epsilon - \epsilon^-]), \dot{\epsilon} > 0 \\ \sigma - \sigma^+ &= 2f(\tfrac{1}{2}[\epsilon - \epsilon^+]), \dot{\epsilon} < 0 \end{aligned}\right\} . \tag{2}$$

Here, (ϵ^-, σ^-) and (ϵ^+, σ^+) are the points of strain and stress respectively at the applicable sign reversal of stress rate (or equivalently, strain rate).

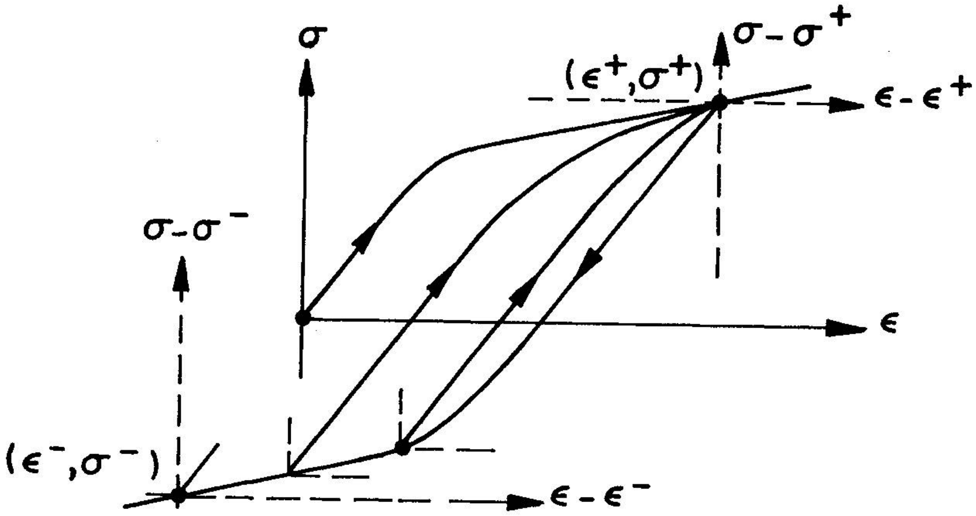

FIG. 1. Uniaxial stress-strain curve and steady-state hysteresis loops

In cyclic proportional deformation these four parameters are constant after the first cycle. Point (ϵ^-, σ^-) satisfies Eq. 2b, so that two scalars are sufficient to characterize the stress strain cycle. Although this model is highly idealized since it gives a steady loop after just one cycle, it does provide a useful basis for design or for more accurate analysis of cyclic creep or relaxation phenomena.

We generalize this description to three-dimensional stress states, assuming that the principal stresses are proportional and remain constant in direction. Define a suitable equivalent strain function

$$\epsilon = \epsilon(\epsilon_1, \epsilon_2, \epsilon_3), \tag{3}$$

symmetric in the indices and specializing to $\epsilon = \epsilon_1$ if $\epsilon_2 = \epsilon_3 = 0$; if we calculate the strain rate $\dot{\epsilon}$ from this equation, the constitutive relations for three-dimensional states can be written formally as Eqs 1 and 2, with σ and ϵ now denoting vectors ($\sigma = \sigma_i$, $i = 1, 2, 3$ etc.) and f denoting a vector function. These constitutive relations are piecewise finite, and for cyclic loading characterized completely by the functions f_i, and two "end" points of the stress path σ_i^+ and σ_i^- (or, equivalently, ϵ_i^- and ϵ_i^+). This extends the Masing relationship (between Eq. 1 and 2) to cyclic proportional triaxial deformation.

3. SELF MAPPING OF BOUNDARY VALUE PROBLEMS

Consider the following class P(1,1) of boundary value problems (of geometrically linear elastostatics):

$$\begin{aligned} &A\sigma + X = 0, \quad B\delta = \epsilon, \quad \sigma = F(\epsilon) \quad \text{in } V; \\ &N\sigma = T \quad \text{in} \quad \partial V' \subset \partial V; \\ &M\delta = D \quad \text{in} \quad \partial V - \partial V'; \end{aligned} \tag{4}$$

where A, B, N and M are linear (differential) operations; V is the region, bounded by ∂V, occupied by the body in the unstressed state; δ, ϵ, and σ are vector-valued point functions in V (representing displacement, strain and stress respectively); and X, T, and D are prescribed functions (representing body force, surface traction and prescribed displacement respectively). Let a progression $P^o(\lambda)$ of boundary values in class P(1,1) be defined by

$$[X,T,D] = [\lambda X^o, \lambda T^o, \lambda D^o], \tag{5}$$

where X^o, T^o, D^o are constant. Assume that a unique solution exists, viz. the progression

$$[\delta, \epsilon, \sigma] = [\delta^o(\lambda), \epsilon^o(\lambda), \sigma^o(\lambda)]. \tag{6}$$

Now consider the related class $P(\frac{1}{2}, \frac{1}{2})$ of boundary values obtained by replacing the constitutive relation $\sigma = F(\epsilon)$ in Eq. 4 with

$$\sigma/2 = F(\epsilon/2), \tag{7}$$

and let a new progression $P(\mu)$ of boundary value problems be defined by

$$[X, T, D] = [2\mu X^o, 2\mu T^o, 2\mu D^o]. \tag{8}$$

Then, $P(\mu)$ has the solution

$$[\delta, \epsilon, \sigma] = [2\delta^o(\mu), 2\epsilon^o(\mu), 2\sigma^o(\mu)], \tag{9}$$

as is easily verified.

Now, let Eq. 5, with λ increasing from 0 to λ^+ represent the loading cycle for a body governed by Eq. 4 with constitutive relation F as in Eq. 1; the ensuing displacements, strains and stresses are given by Eq. 6. When the load factor reverses, the corresponding values are given as

$$[\delta^+, \epsilon^+, \sigma^+] \equiv [\delta^o(\lambda^+), \epsilon^o(\lambda^+), {}^o(\lambda^+)] \tag{10}$$

If we make the substitutions

$$\delta - \delta^+ \rightarrow \delta, \sigma - \sigma^+ \rightarrow , \epsilon - \epsilon^+ \rightarrow \epsilon, \tag{11}$$

the boundary value problem progression for increments of stress, strain and displacement for the unloading cycle with λ decreasing from λ^+ to λ^- is generated by Eq. 8 with μ representing $\lambda - \lambda^+$. Eq. 9 gives the solution. The unloading path for all points of the body is determined by the corresponding points on the loading path. We conclude: If a material under cyclic proportional triaxial deformation follows the Masing relationship, the stresses, strains and deformations at all points in a body made of this material, subjected to cyclic proportional loading, will also follow a Masing relationship.

4. SIMPLE MATERIALS

A convenient approximation to the uniaxial hardening curve is the power law

$$\sigma = c\epsilon^{(1/n)} \equiv c(\text{sign}\epsilon)|\epsilon|^{1/n} \tag{12}$$

where c and n are positive material constants. The exponent in parenthesis is a convenient symbolic notation which reduces to an ordinary exponent if n is an odd integer.

We may generalize this to complex stress states as follows. Denote the principal shearing stresses and corresponding strains respectively by

$$\tau_i = \tfrac{1}{2}|\sigma_j - \sigma_k| \quad ; \quad \gamma_i = \tfrac{2}{3}|\epsilon_j - \epsilon_k| \tag{13}$$

where i, j, k is any permutation of 1,2,3.

Define the plastic potential as

$$W = \frac{2^n c}{n+1}\left[\tau_1^{(n+1)} + \tau_2^{(n+1)} + \tau_3^{(n+1)}\right] \; ; \tag{14}$$

Using Eq. 13 this gives the strains

$$\epsilon_1 = \frac{c}{2}\left[(\sigma_1 - \sigma_2)^{(n)} - (\sigma_3 - \sigma_1)^{(n)}\right] \tag{15}$$

and the analogous expressions by cyclic permutation.

Assume that for a certain value of $\tau_3 = \tau_{3A} = \frac{1}{2}(\sigma_{1A} - \sigma_{2A})$ the further stress path corresponds to a change of sign of $\dot{\tau}_3$ from positive to negative while $\dot{\tau}_2$ and $\dot{\tau}_1$ remain positive. Then instead of Eq. 14 the stress potential takes the form

$$W = \frac{2^n c}{n+1} [\tau_1^{(n+1)} + \tau_2^{(n+1)} + (\tau_3 - \tau_{3A})^{(n+1)}] \tag{16}$$

yielding readily the stress-strain relations for the new path.

Alternatively we may derive a simplified hardening model already discussed by Mróz [2]. The state of hardening is assumed to be described with sufficient accuracy in terms of a set of surfaces of constant hardening moduli $K = \partial\sigma_f/\partial\varepsilon$, where σ_f denotes the component of the stress increment along the normal to the yield surface and $\varepsilon = (d\varepsilon_{ij} d\varepsilon_{ij})^{\frac{1}{2}}$ denotes the absolute value of the plastic strain increment. Since these surfaces cannot intersect, the active surfaces are assumed to translate with the stress point and become tangential along the stress path. For a piecewise linear yield condition this model results in finite stress-strain relations valid in particular sub-domains of stress space similar to the model described above. Fig. 2 show the translation for two radial stress paths.

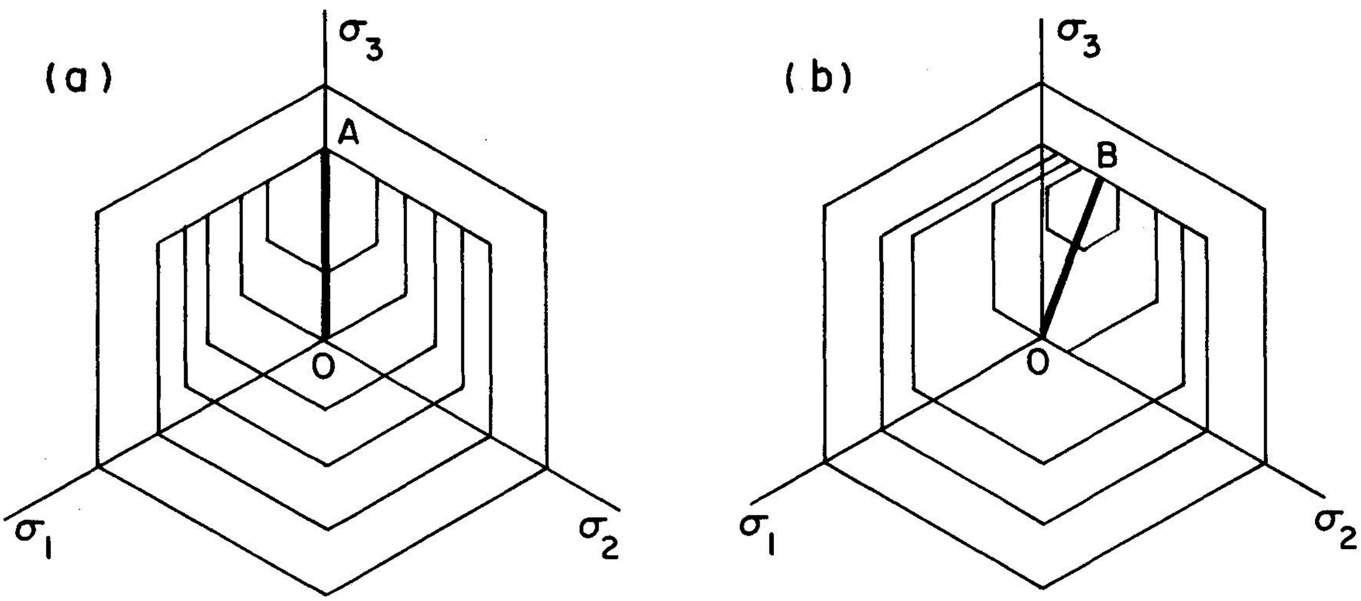

FIG. 2. Fields of hardening moduli after plastic loading
(a) along OA (b) along OB

The postulated materials have piecewise finite stress-strain relations that are homogeneous functions of order n. The materials may be called simple, incrementally hyperelastic of order n [4]. For loading increment $\Delta\lambda$, the corresponding stress increment is everywhere proportional to $\Delta\lambda$ and the corresponding strain and deformation increments are proportional to $\Delta\lambda^{(n)}$. If $[\delta°,\varepsilon°,\sigma°]$ solves $[X°,T°,D°]$, then $[\lambda^{(n)}\delta°,\lambda^{(n)}\varepsilon°,\lambda\sigma°]$ is the solution field for $[\lambda X°,\lambda T°,\lambda^{(n)}D°]$. Thus, the entire solution for a cyclic loading (either traction-controlled: $D° = 0$ or displacement-controlled: $X°,T° \equiv 0,0$) can be derived from a single equilibrium solution. This solution can be obtained for many structures of practical interest, using a variety of numerical methods. We note, in particular that the unloaded state is stress free under cyclic proportional loading.

5. EXAMPLE: THICK-WALLED CYLINDER

Consider a long thick-walled tube of internal and external radii a and b, subjected to internal pressure p varying between the prescribed limits p^+ and p^-. A simple, closed form cyclic solution can be obtained when only plastic strains satisfying Eq. 15 are accounted for. Since the strain normal to the plane of deformation vanishes, we have

$$\epsilon_\theta + \epsilon_r = \frac{u}{r} + \frac{du}{dr} = 0 , \quad u = \frac{A}{r} \tag{17}$$

where u denotes the radial displacement and ϵ_θ, ϵ_r are principal strains in the

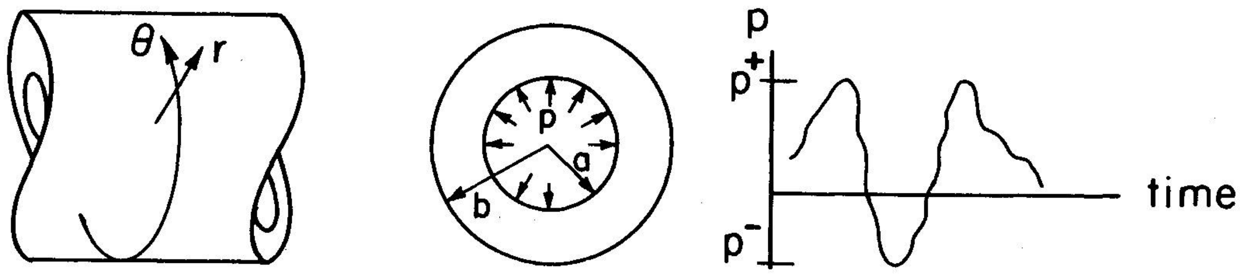

FIG. 3. Thick walled cylinder under cyclic pressure

plane (r,θ) ; A denotes an integration constant. From Eq. 15 we have

$$\epsilon_\theta = C'(\sigma_\theta - \sigma_r)^n, \quad \epsilon_r = -C'(\sigma_\theta - \sigma_r)^n, \tag{18}$$

where $C' = c(1 + 2^{-n})/2$. The inverse relations take the form

$$\sigma_\theta - \sigma_r = C(\epsilon_\theta)^{1/n} = C(A/r^2)^{1/n} , \tag{19}$$

where $C = (C')^{-n}$. Using the equilibrium equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{20}$$

the stress state within the tube is determined in the form

$$(\sigma_r;\sigma_\theta) = \frac{p}{(\frac{b}{a})^{2/n} - 1} \left[(\frac{b}{r})^{2/n} - 1; \quad 1 - (1 - 2/n)(\frac{b}{r})^{2/n} \right] \tag{21}$$

where the boundary conditions $\sigma_r = -p$ for $r = a$ and $\sigma_r = 0$ for $r = b$ have been satisfied. The displacement field is given by

$$u = \frac{2^n p^n b^2}{C^n n^n [(\frac{b}{a})^{2/n} - 1]} \frac{1}{r} . \tag{22}$$

Consider now the unloading program. Denote by $\Delta p = p - p^+$, $\Delta u = u - u^+$
$\Delta\epsilon_r = \epsilon_r - \epsilon_r^+$, $\Delta\epsilon_\theta = -\epsilon_\theta^+$. Instead of Eq. 19 we now have

$$\Delta\sigma_\theta - \Delta\sigma_r = 2C\left(\frac{\Delta\varepsilon_\theta}{2}\right)^{1/n} = D(\Delta\varepsilon_\theta)^{1/n} = D\left(\frac{\Delta A}{r}\right)^{1/n} , \quad (23)$$

where $D = 2^{\left(\frac{n-1}{n}\right)} C$. The equilibrium equations now provide expressions for stresses identical to Eq.21 with p replaced by Δp. The radial displacement for any p satisfying $p^- \leq p < p^+$ equals

$$u_2 = \frac{b^2 p^n}{C^n n^n [(\frac{b}{a})^{2/n} - 1]} (2^n - 2) \quad (24)$$

Since the stress state does not depend on constants D or C, upon removal of the pressure both stresses vanish. Thus, no residual stresses are created for zero pressure in the steady state. For further repetition of pressure between p^+ and p^-, the displacement and stress fields are described by Eqs. 21 and 24.

6. CONCLUSION

Plastic analysis of practical structures under variable loading, however difficult in general, is tractable for cyclic proportional loading when the plastic behaviour of the material can be adequately described by a Masing-type relationship with an incremental power-law, and when one solution to the corresponding static nonlinear elastic boundary value problem can be produced. An example is given herein; other examples (circular and annular plates, etc.)have been presented by the authors elsewhere [4]. Experimental verification of the practical validity of such analysis is currently underway.

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SUMMARY

Piecewise finite representations of material behaviour are proposed for practical analysis of plastic metal behaviour under cyclic loading.

RESUME

Quelques représentations constitutives finies du matériau sont présentées pour l'analyse pratique du comportement plastique des structures sous charges cycliques.

ZUSAMMENFASSUNG

Stückweise endliche Modelle für das Stoffverhalten werden zur praktischen Berechnung zyklisch beanspruchter plastifizierender Metallkonstruktionen vorgeschlagen.