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Ellipses of Inertia for Nonlinear Analysis of Structures

Ellipses d'inertie pour l'analyse non-linéaire des structures

Trägheitsellipsen für die nichtlineare Berechnung von Konstruktionen

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SUMMARY
A general procedure is presented for the analysis of concrete structures composed of linear members, which is based on a particular definition of the structural stiffness. This is derived from a linearization of the paths, run by all the points of the materials, on their respective constitutive law for a given loading step.
Modified Culmann's ellipses of inertia of the corss sections allow exact incorporation of this idealization into the stiffness matrix of the structure.
The method is well suited for taking into account the interaction of the internal forces correctly. Examples of applications are shown, including monotonic and cyclic loadings.

RÉSUMÉ
Une procédure générale est présentée pour l'analyse de structures en béton composées d'éléments monodimensionnels, basée sur une définition particulière de la rigidité de la structure. Cette rigidité est derivée d'une linéarisation des chemins, parcourus par tous les points des matériaux sur leurs propres lois tension-déformation, pour un incrément donné des charges. Des ellipses de Culmann modifiées des sections droites permettent d'introduire exactement ce schéma dans la matrice de rigidité de la structure.
La méthode est très apte à prendre en compte correctement l'intéraction des efforts. Des exemples d'application sont présentés, comprenant charges monotones et cycliques.

ZUSAMMENFASSUNG
Ein allgemeines Verfahren für die Analyse von aus Stabelementen bestehenden Konstruktionen wird erläutert, das auf einer besonderen Definition der Steifigkeit beruht. Diese wird abgeleitet aus einer Linearisierung der Wege aller Punkte der Materialen in Bezug auf ihr Spannungs-Dehnungsverhältnis für eine gewisse Belastungsstufe.
Modifizierte Culmann's Trägheitsellipsen der Querschnitte gestatten, dieses Schema in die Steifigkeits-Matrix der Konstruktionen einzuführen.
Diese Methode eignet sich besonders, um die Interaktion innerer Kräfte korrekt in Betracht zu nehmen; Anwendungsbeispiele werden gezeigt, monotone und zyklische Belastungen inbegriffen.

1. INTRODUCTION

The basic factors generating an overall nonlinear response of the structure are geometric factors, important for slender compressed members, and mechanical factors.

Due to them, the stiffness (or the deformability) of the structure is altered following the gradual application of the loadings, and all the internal forces redistribute continuously. An accurate description of the varying stiffness of the structure is essential for the correct solution of the problem.

The numerical reproduction by matrix analysis of the behavior of structures requires a linearization of the stiffness corresponding to a given increment of the loadings or of the deformations.

The ways of performing the linearization, i.e. of defining the stiffness coefficients satisfying the correct relationships of nodal forces and displacements, are infinity; thus the actual selection is arbitrary. Among all the possible ways, the one is assumed here corresponding to a particular pseudo--elastic criterion which is deemed to be the most valuable, and conceptually correct.

In case of beam flexural elements, this criterion leads to a synthetic and easy procedure for building up the stiffness matrix, taking advantage of Culmann's theory of the ellipse of inertia.

2. IDEALIZATION OF THE STRUCTURAL PROBLEM

A set of idealizations, including a discretization of the structure into elements, is necessary in order to establish the model for the analysis.

The analysis of concrete beam elements is generally performed by first considering several representative cross sections, an then working out the characteristics of the whole element.

For the cross section, usually the linear distribution of strains is assumed, including the bond between concrete and steel. Various models of stress-strain relationships are used for the materials, either holonomic or including stress reversal paths.

For the element, the distribution of the deformabilities, within the segment connecting the analyzed sections, may be assumed as varying linearly, or with different laws. A proper selection of the spacing of sections must be made, in order to realize a suitable equivalent length of plastic rotation zones, according to empirical formulas.

The effects of cracking and tension stiffening of concrete is often accounted for approximately, through a conventional constitutive law. Shear deformations are disregarded, in dealing with flexural structures.

Once a consistent idealization is set down for the structure and for the external actions, the analytical problem is defined.

The solution may be obtained by various procedures, that will be generally iterative, due to the nonlinearity. The stiffness or the deformability method may be employed. The forces may be applied either at once or step by step.

Taking for instance the most general case of a stiffness method step by step procedure, the solution of a set of equations like

$$\mathbf{K}_{st} \cdot \Delta\mathbf{U} = \Delta\mathbf{P} \tag{1}$$

is to be performed at each step; where

$\mathbf{K}_{st}$ is a stiffness matrix of the structure

$\Delta\mathbf{U}$ is the vector of nodal displacements

$\Delta\mathbf{P}$ is the vector of applied nodal forces;

all being referred to the actual step.

$\mathbf{K}_{st}$ is only valid for the particular step and for the particular set of loads, as it represents a linearization of a nonlinear relationship.

The criterion of linearization is deemed to be of major importance for finding the comprehensive correct solution, and for a speedy convergence.

The criterion is arbitrary. In fact, the matrix $\mathbf{K}_{st}$ is n x n (n being the number of components of $\Delta\mathbf{U}$), so that infinite combinations of its coefficients satisfy eqn. (1), given $\Delta\mathbf{U}$ and $\Delta\mathbf{P}$.

The criterion presented in the following has been employed for numerous applications and allows for a detailed analysis of the behavior of any structure idealized as above.

3. STIFFNESS CRITERION

The stiffness matrix of the structure will be derived from a secant linearization of the paths run by "every" point of material in the structure on its own constitutive law, during the considered step.

Selected representative cross sections are suitably discretized into a finite number of areolas of the different materials (Fig. 1).

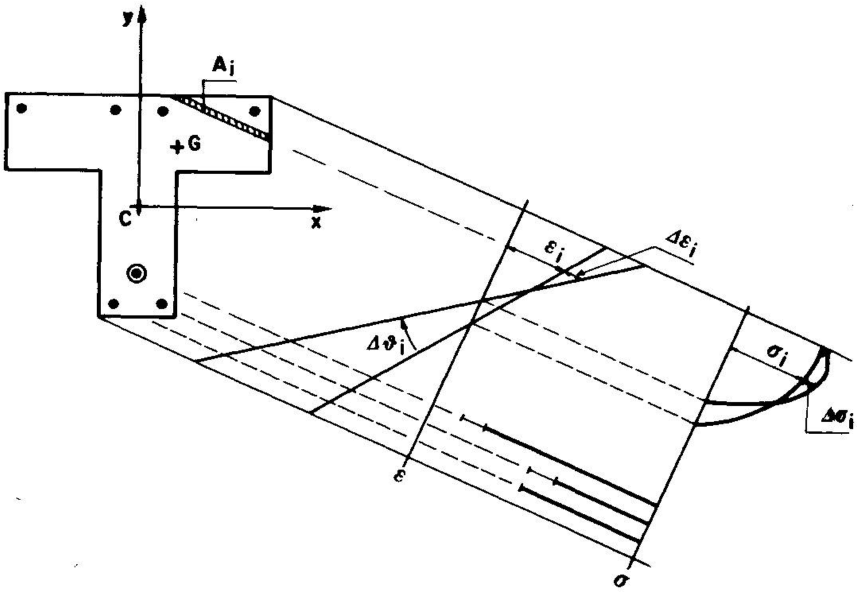

Fig. 1

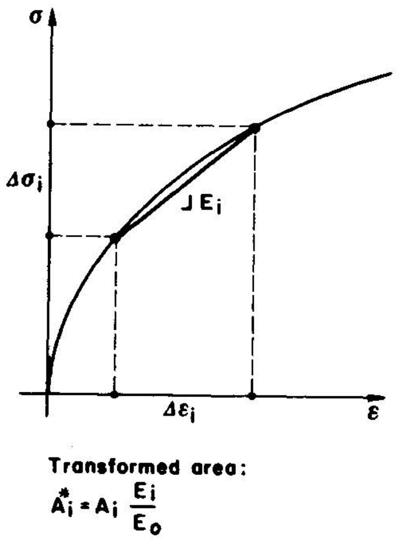

Fig. 2

If $\Delta\epsilon_i$ and $\Delta\sigma_i$ are the strain and the stress variations undergone by the i-th areola during the step (Fig. 2), a pseudo-elastic modulus is defined as:

$$E_i = \frac{\Delta\sigma_i}{\Delta\epsilon_i} \tag{2}$$

Then the areola is assigned a transformed area:

$$A_i^* = A_i \frac{E_i}{E_0} \tag{3}$$

A_i being its geometric area, and E_0 an arbitrary reference modulus, which can be taken as $E_0 = 1$.

This operation has the most general application: the stress-strain state of the areolas may run on strain-softening branches (E_i negative) or rest on a cracked part ($E_i = 0$, if the crack remains opened for the step) or run on strain reversal paths of any shape.

Extending the calculation of A_i^* to all the areolas of the section, a whole transformed section is obtained. This can be handled as a homogeneous linear elastic section, having a modulus of elasticity E_0.

Namely, all the typical characteristics of the homogeneous sections, as Centroid G, Area A, Static Moments S, Moments of Inertia J, etc, may be easily calculated. Indeed, the well known Culmann's ellipse of inertia [1] may be built up, which summarizes all these characteristics and relates the internal forces N M_x M_y with the strain state (except for the E modulus).

When putting $E_0 = 1$, such a modified ellipse not only relates the load center with the neutral axis, but it completely represents the secant stiffness or deformability of the cross section for the step.

Of course, it is meaningful for the original section, only when referring to the particular forces applied at the particular step, due to eqn (2), and it may be called the "step secant" ellipse of the section.

But its usefulness is twofold. First, the ellipse is very helpful in finding the stress-strain state on the section itself, corresponding to a given internal forces variation. Then, it is necessary for the construction of the structure's stiffness relationships according to the criterion adopted.

4. CROSS SECTION ANALYSIS

During the iterative procedure for solving the structure's problem (eqn. 1), at each cycle all the cross sections must be analyzed to find their stress strain variation, as well as the corresponding secant deformability, expressed by the relationship

$$\Delta \mathcal{E} = \mathbf{K}_{cs}^{-1} \cdot \Delta \mathbf{S} \tag{4}$$

where:

$\Delta \mathcal{E} \equiv \{\Delta\epsilon_c, \Delta\theta_x, \Delta\theta_y\}$ is the vector of deformation variations: strain of reference point C, and curvatures about the fixed axes x, y, (Fig. 1);

$\Delta \mathbf{S} \equiv \{\Delta N, \Delta M_x, \Delta M_y\}$ is the vector of force variations; and

$\mathbf{K}_{cs}^{-1}$ is the deformability matrix of the cross section, that is to be found together with $\Delta\mathcal{E}$; it will be defined as follows:

$$\mathbf{K}_{cs}^{-1} \equiv \begin{vmatrix} \left(\dfrac{\partial\epsilon_c}{\partial N}\right)^* & \left(\dfrac{\partial\epsilon_c}{\partial M_x}\right)^* & \left(\dfrac{\partial\epsilon_c}{\partial M_y}\right)^* \\ \left(\dfrac{\partial\theta_x}{\partial N}\right)^* & \left(\dfrac{\partial\theta_x}{\partial M_x}\right)^* & \left(\dfrac{\partial\theta_x}{\partial M_y}\right)^* \\ \left(\dfrac{\partial\theta_y}{\partial N}\right)^* & \left(\dfrac{\partial\theta_y}{\partial M_x}\right)^* & \left(\dfrac{\partial\theta_y}{\partial M_y}\right)^* \end{vmatrix} \tag{5}$$

and directly written down throughout the parameters of the modified ellipse of inertia of the section (Fig. 3).

$$\begin{aligned}
\left(\frac{\partial\epsilon_c}{\partial N}\right)^* &= \frac{1}{A} + \frac{\xi_c^2}{J_\xi} + \frac{\eta_c^2}{J_\eta} \\
\left(\frac{\partial\epsilon_c}{\partial M_x}\right)^* &= \frac{\xi_c}{J_\xi}\cos\tau + \frac{\eta_c}{J_\eta}\sin\tau = \left(\frac{\partial\theta_x}{\partial N}\right)^* \\
\left(\frac{\partial\epsilon_c}{\partial M_y}\right)^* &= \frac{\xi_c}{J_\xi}\sin\tau + \frac{\eta_c}{J_\eta}\cos\tau = \left(\frac{\partial\theta_y}{\partial N}\right)^* \\
\left(\frac{\partial\theta_x}{\partial M_x}\right)^* &= \frac{1}{J_\xi}\cos^2\tau + \frac{1}{J_\eta}\sin^2\tau \\
\left(\frac{\partial\theta_x}{\partial M_y}\right)^* &= \left(\frac{1}{J_\eta} - \frac{1}{J_\xi}\right)\sin\tau\cos\tau = \left(\frac{\partial\theta_y}{\partial M_x}\right)^* \\
\left(\frac{\partial\theta_y}{\partial M_y}\right)^* &= \frac{1}{J_\xi}\sin^2\tau + \frac{1}{J_\eta}\cos^2\tau
\end{aligned} \tag{6}$$

The expressions of the above coefficients require some observations.

The areas and moments of inertia in eqns. (6) are not only geometric, but contain the moduli of elasticity too, having put $E_0 = 1$ in eqn. (3).

The partial derivatives appearing in eqns. (5) and (6) are marked with a star (*), meaning that they are calculated on the pseudo-elastic secant scheme: for instance, $(\partial\epsilon_c/\partial N)^*$ is not calculated as variation of ϵ_c for a variation of N alone, but for a variation of N on the transformed section, which accounts for the actual interaction of ΔM_x and ΔM_y with ΔN for that step.

The matrix appears symmetric, due to the pseudo-elastic homogenization.

The way of calculating the coefficients (eqns. 6) corresponds to the particular definition given for the secant deformability.

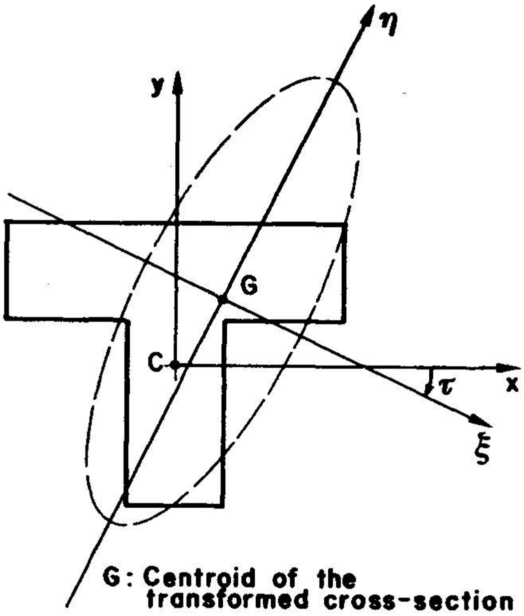

Fig. 3 — Ellipse of inertia of the transformed section.

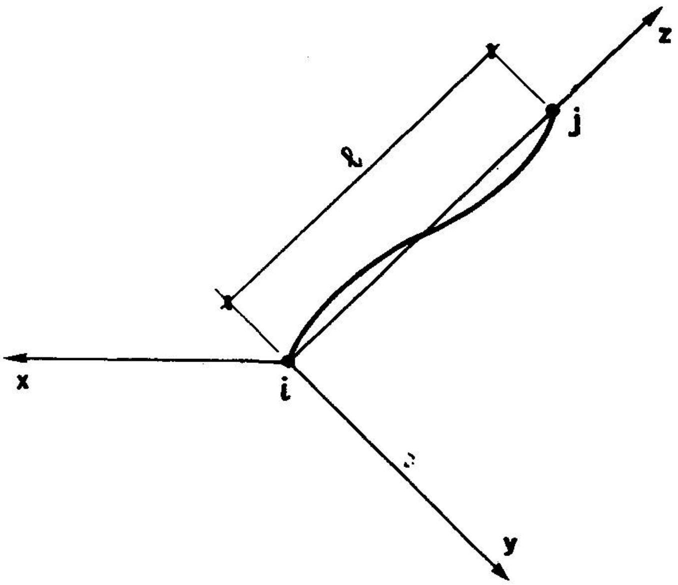

Fig. 4 — Intrinsic coordinates of the element i - j.

During the iterative solution of eqn. (4), the matrix K_{cs}^{-1}, i.e., the modified ellipse of inertia of the cross section, is easily built up at every cycle, as follows.

The section is discretized into a large number of areolas of concrete, of ordinary and of prestressing steel (initial prestress is accounted for on all the materials). From a first tentative strain variation, all the transformed areas A_i^* are calculated by means of eqns. (2) and (3), as well as all the characteristics A, J etc of the transformed section. These, introduced into K_{cs}^{-1} through eqns. (6), allow for a new vector $\Delta\mathcal{E}$ being calculated, as function of the given $\Delta \mathbf{S}$ (eqn. 4). The cycle is repeated up to convergence.

This is sped up substantially by the step secant ellipse, which provides immediately good orientation toward the state of stress produced by the combined internal forces $\Delta\mathbf{S}$.

On the other hand, it must be said that this procedure is comparatively less effective when considering only the gradual softening of the material for increasing stress. In fact, in cases of single or independent variables, as for pure compression or pure uniaxial bending, procedures like Newton-Raphson are faster in converging.

But, when concrete sections are subject to combined forces, the ordinary partial derivatives would be extremely (and uncorrectly) sensitive to the position of the neutral axis and, specially, to its deviation induced by finite variations of internal forces. Then, the much better orientation of the derivatives defined as in eqns. (6) becomes the decisive factor.

Furthermore, it was shown that this procedure offers the simultaneous finding of the correct secant deformability itself of the cross section, K_{cs}^{-1}, which represents the best basis for the definition of the stiffness of the element and of the whole structure.

5. ELEMENT AND STRUCTURE STIFFNESS

The problem of selecting a proper linearization of the stiffness is quite the same as for the cross section; only there are more interacting forces.

To define the mechanic stiffness of an element i-j in its intrinsic coordinate system (Fig. 4), five components are to be related:

$$\Delta \mathbf{F} = \mathbf{K}_{el} \cdot \Delta \boldsymbol{\Phi} \tag{7}$$

where:

$$\Delta \mathbf{F} \equiv \{\Delta N, \Delta M_{xi}, \Delta M_{xj}, \Delta M_{yi}, \Delta M_{yj}\}$$

$$\Delta \boldsymbol{\Phi} \equiv \{\Delta \Delta l, \Delta \phi_{xi}, \Delta \phi_{xj}, \Delta \phi_{yi}, \Delta \phi_{yj}\}$$

$\mathbf{K}_{el}$ is the stiffness matrix of the element, yet to be defined.

As, for the section, the deformability was worked out by considering the areolas of material as linear elastic consistently with the secant moduli; thus, for the element, the cross section will be considered linear elastic consistently with the secant ellipses.

Thus, the definition of $\mathbf{K}_{el}$ consists in expressing the stiffness of an elastic beam with variable cross section, which can be done by means of well known procedures.

First, the deformability matrix is built up:

$$\mathbf{K}_{el}^{-1} \equiv \left| \begin{array}{ccccc} \left(\frac{\partial \Delta l}{\partial N}\right)^* & \left(\frac{\partial \Delta l}{\partial M_{xi}}\right)^* & \left(\frac{\partial \Delta l}{\partial M_{xj}}\right)^* & \left(\frac{\partial \Delta l}{\partial M_{yi}}\right)^* & \left(\frac{\partial \Delta l}{\partial M_{yj}}\right)^* \\ & \left(\frac{\partial \phi_{xi}}{\partial M_{xi}}\right)^* & \left(\frac{\partial \phi_{xi}}{\partial M_{xj}}\right)^* & \left(\frac{\partial \phi_{xi}}{\partial M_{yi}}\right)^* & \left(\frac{\partial \phi_{xi}}{\partial M_{yj}}\right)^* \\ & & \left(\frac{\partial \phi_{xj}}{\partial M_{xj}}\right)^* & \left(\frac{\partial \phi_{xj}}{\partial M_{yi}}\right)^* & \left(\frac{\partial \phi_{xj}}{\partial M_{yj}}\right)^* \\ & \text{(Symm)} & & \left(\frac{\partial \phi_{yi}}{\partial M_{yi}}\right)^* & \left(\frac{\partial \phi_{yi}}{\partial M_{yj}}\right)^* \\ & & & & \left(\frac{\partial \phi_{yj}}{\partial M_{yj}}\right)^* \end{array} \right| \tag{8}$$

The starred derivatives have the same meaning as in cross section analysis, i.e., they are calculated on the secant pseudo-elastic scheme; therefore, the matrix in eqn. (8) is also symmetric.

The coefficients are calculated by numerical integration of the deformed axis line, through the coefficients of the matrixes $\mathbf{K}_{cs}^{-1}$ of the selected cross sections in the element, according to the distribution assumed for the deformabilities between two of them.

$$\begin{aligned} \left(\frac{\partial \Delta l}{\partial N}\right)^* &= \int_0^l \left(\frac{\partial \epsilon_c}{\partial N}\right)^* dz \\ \left(\frac{\partial \Delta l}{\partial M_{xi}}\right)^* &= \int_0^l \left(\frac{\partial \epsilon_c}{\partial M_x}\right)^* \left(\frac{z}{l} - 1\right) dz \\ \left(\frac{\partial \Delta l}{\partial M_{xj}}\right)^* &= \int_0^l \left(\frac{\partial \epsilon_c}{\partial M_x}\right)^* \frac{z}{l}\, dz \end{aligned} \tag{9}$$

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Finally, by inverting the matrix $\mathbf{K}_{el}^{-1}$, the stiffness matrix of the element, $\mathbf{K}_{el}$, is obtained.

Then, the stiffness matrix $\mathbf{K}_{st}$ of the structure is built up, as for a linear elastic structure: i.e. the intrinsic stiffness matrixes $\mathbf{K}_{el}$ are projected onto the global coordinate system – eventually being added "geometric matrixes" – and are assembled to form $\mathbf{K}_{st}$.

This matrix is adjusted iteratively during a step, up to convergence of eqn. (1).

It has the property, like the section and the element matrixes seen above, of incorporating exactly the secant stress-strain paths of all the considered points. Therefore it offers a very accurate relationship between the set of external forces and the displacements.

Indeed, all its coefficients represent partial derivatives calculated on the secant elasticity resulting from the combined acting of all the components of $\Delta\mathbf{P}$ (eqn. 1); the meaning of the generic coefficient of $\mathbf{K}_{st}$ is in fact:

$$k_{mn} = \left(\frac{\partial P_m}{\partial U_n}\right)^* \tag{10}$$

(where the asterisk is to be understood as before).

As a consequence, the internal forces distribution rapidly assumes the correct trim in the iterations. Whereas the calculation of derivatives from the separate application of the loading components, i.e.,

$$k_{mn} = \frac{\partial P_m}{\partial U_n} \tag{11}$$

would yield very dispersive iterations, particularly when the degrees of freedom are numerous, and the mutual orientation of the displacements is much affected by the simultaneous action of various forces.

6. TYPICAL APPLICATIONS

The described ellipse was called "step-secant", being referred to the secant elasticity of the structure during a step of an incremental analysis.

Incremental analysis is convenient in several cases, as for example in highly hyperstatic structures showing subsequent formations of "plastic hinges"; while it necessary in cases of loading reversals, or even of monotonic loadings inducing significant stress reversals. Some applications have been done, with computer programs using this type of analysis [2, 4].

Figures 5, 6, 7 illustrate the results of the analysis of a micro-concrete multistory frame for reproducing an experimental test. The calculated load-deflection cyclic curve matched the test result very well. Moreover, the moment-curvature history under variable normal force was recorded in all the sections as shown in Fig. 6.

In other cases, the use of different types of linearization may be suitable: the "full-secant" linearization and the "tangent" linearization, which are limit cases of the previous one and require the same sequence of calculation.

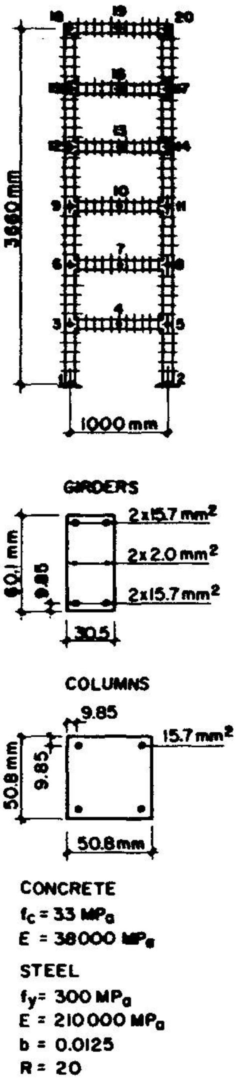
An example is the case of slender bridge piers in biaxial bending [3]. These structures are generally statically determinate or lowly indeterminate, and do not undergo extended plasticizations. On the other hand, the variability of the cross section and the presence of distributed normal and lateral forces produce important relative deviations of the neutral axis along the height of the structure (Fig. 9).

Then, the design loads may be applied in a single step, the pseudo-elastic model resulting full--secant. Eqn. (2) becomes

$$E_i = \frac{\sigma_i}{\epsilon_i} \tag{12}$$

The deformed axis line is numerically integrated along the height, and compatibility conditions are imposed at the top, if the structure is hyperstatic.

In the latter case, the deformability corresponding to small variations of the stress strain state, around the calculated state, must be defined. Therefore a "tangent" ellipse is built up at every cross



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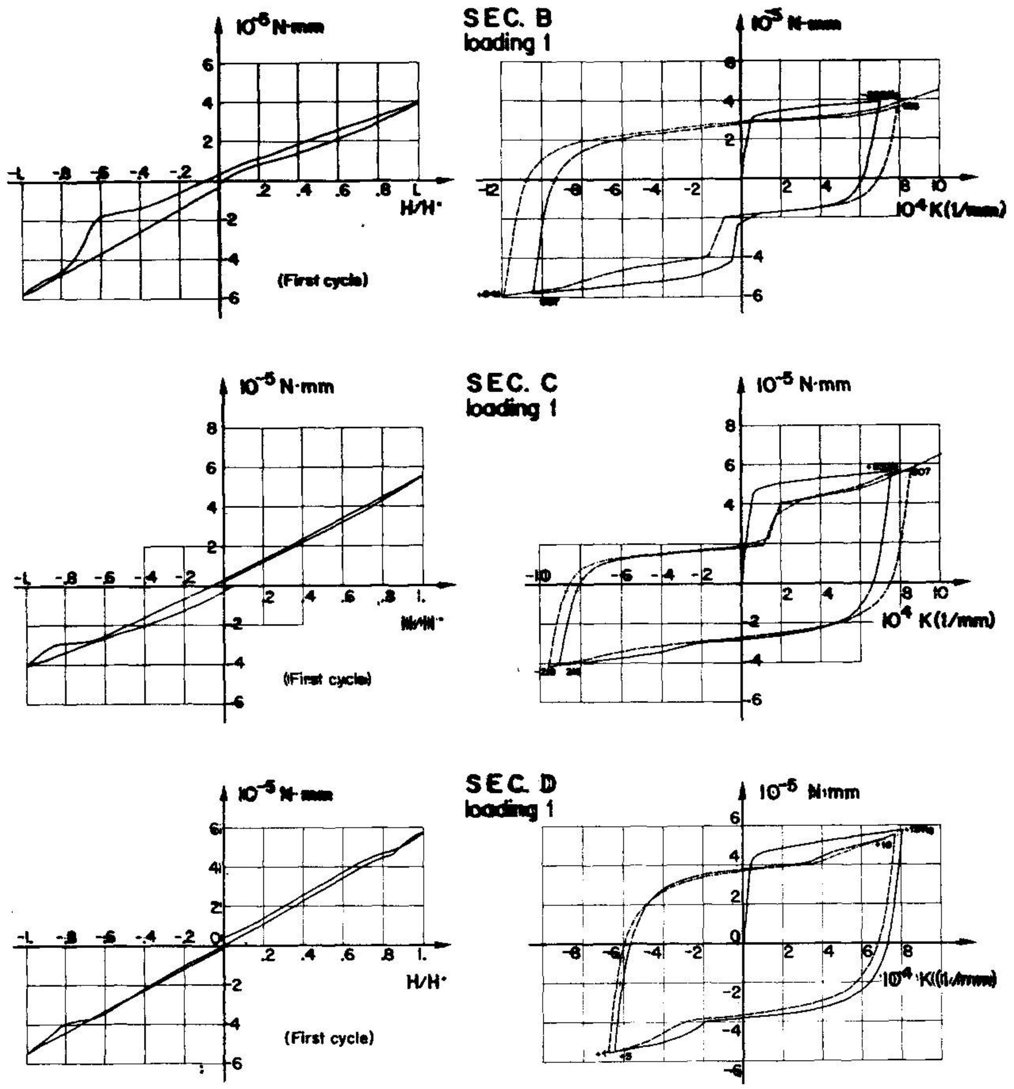

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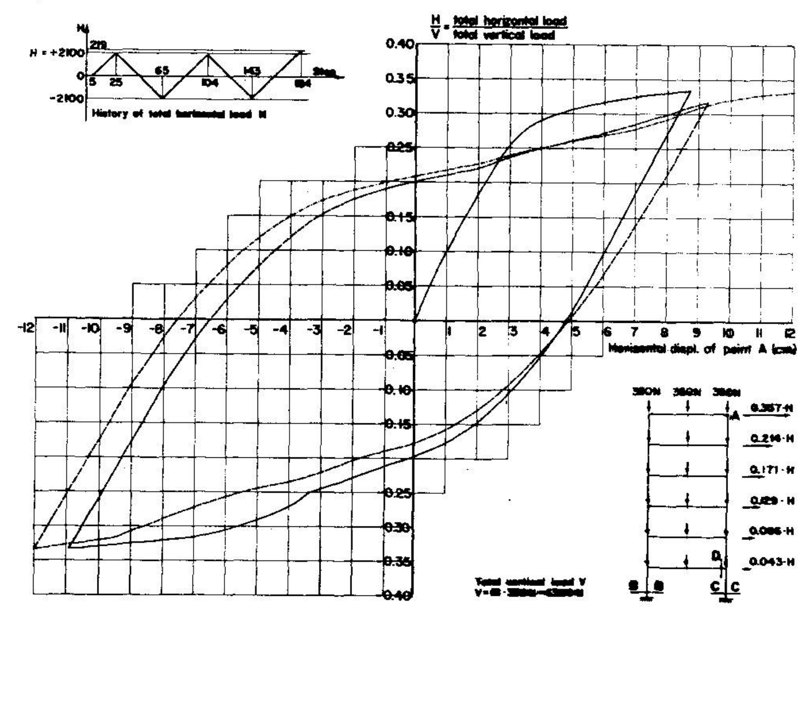

Fig. 5 – Multistory microconcrete frame as described for the analysis [2].

Fig. 6 – Moment-External Load Diagrams and Moment-Curvature Diagrams.

Fig. 7 – Load-displacement diagram obtained analytically.

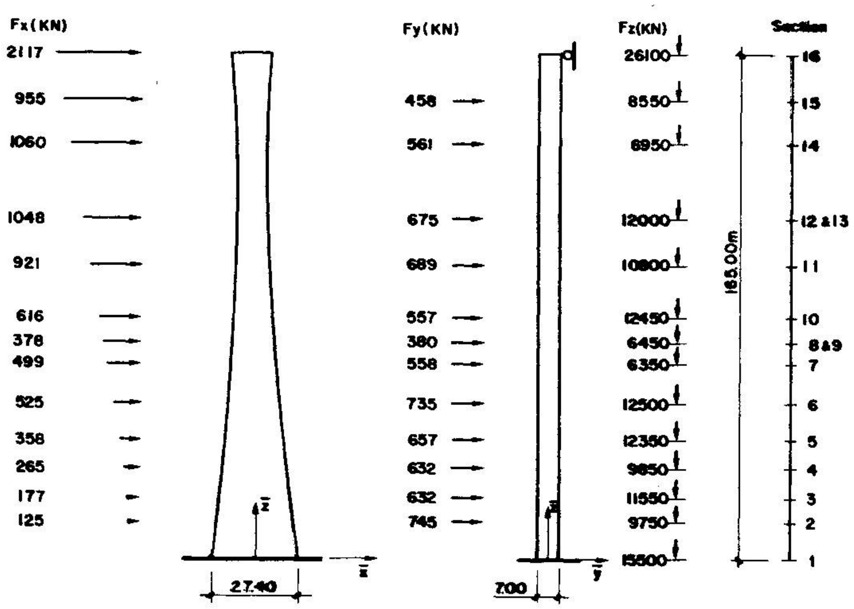

Fig. 8 — Prestressed tall bridge pier, with discretized design loads and cross-sections [3].

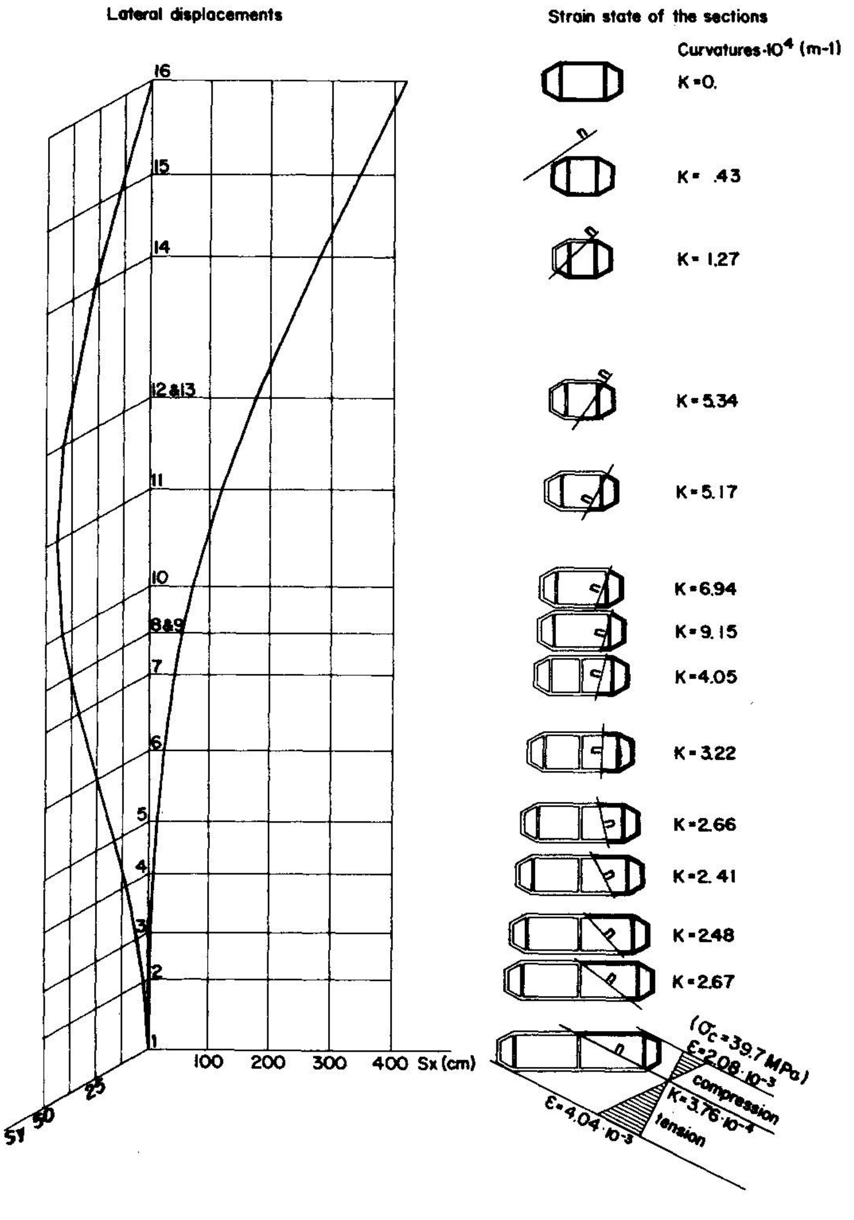

Fig. 9 — Deformation and strain state of the pier in fig. 8 at collapse.

section, using, as pseudo-elastic moduli E_i, the tangent values

$$E_i = \left(\frac{d\sigma_i}{d\epsilon_i}\right)_{S_i} \tag{13}$$

calculated at the strain states S_i of the respective areolas.

These ellipses are used to make up, by integration along the axis of the structure (similar to that of eqn. 9), the flexibility matrix referred to the displacements at the top.

Fig. 8 shows the problem of a prestressed concrete tall bridge pier under triaxial loadings [3]. In Fig. 9 the complete state of stress at collapse is illustrated, as furnished by the analysis.

7. CONCLUSION

The nonlinear behavior of concrete structures is strongly affected by the interaction of and the continuous redistribution of the internal forces.

A general procedure for structural analysis has been presented, based on a particular definition of the linearized stiffness of a structure.

The stiffness matrix of the structure is worked out on the assumption that the materials in every point of the structure — instead of describing the actual curved paths on the respective constitutive laws — describe the corresponding secant segments.

Such a matrix obviously yields, for the whole step of calculation, the same results that would be obtained following the actual nonlinear paths, thus being a correct linearization, as well as any arbitrary ***a posteriori*** linearization fulfilling the true solution. But it is the only one that would yield exact solutions for any fraction of the step, if the local secant paths were true.

Therefore it should be acknowledged as the only true secant linearization for the structural stiffness.

The construction of such a matrix is possible by the use of modified ellipses of inertia of the cross sections, which are the means of incorporating the ***spot*** secant stiffness of the material into the global stiffness relationship. By that means, the local phenomena contributing to the deformation (cracking, yielding, degradation) can be exactly related with all the individual displacement components — provided that they are suitably incorporated in the constitutive laws — theoretically with any degree of precision, if the geometric discretization and the loading steps are fine enough.

The method proved itself to be very effective in the analysis of nonlinear problems of structures, subject to monotonic or cyclic loadings.

Several computer programs have been prepared following it, and many applications have been performed like in the examples sketched in this paper.

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