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# On the Uniform Approximation of Analytic Functions by Means of Interpolation Polynomials

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## 1. Introduction

Let  $D$  be a domain of the complex plane bounded by a smooth closed Jordan curve  $\Gamma$ .

Let  $\{z_k^{(n)}\}$ ,  $1 \leq k \leq n$ ,  $n=1, 2, 3, \dots$  be a triangular matrix of points of  $\Gamma$  ( $z_i^{(n)} \neq z_k^{(n)}$  for  $i \neq k$ ), and consider the fundamental polynomials of the Lagrange interpolation:

$$l_j^{(n)}(z) = \prod_{k \neq j} \frac{z - z_k}{z_j - z_k} \quad (1.1)$$

which have the property that  $l_j(z_j) = 1$  and  $l_j(z_k) = 0$  for  $k \neq j$ . We will say that  $\{z_k^{(n)}\}$  is a *regular point system* if the polynomials  $l_j^{(n)}(z)$  are uniformly bounded in  $\bar{D}$ , i.e. if

$$|l_j^{(n)}(z)| \leq M \quad \text{for } z \in \bar{D}, \quad 1 \leq j \leq n, \quad n = 1, 2, \dots \quad (1.2)$$

for some  $M$ . In particular a system of *Fekete points*<sup>1)</sup> is a regular point system; in fact in this case (1.2) holds with  $M=1$ .

If  $f(z)$  is a function regular in  $D$ , and continuous in  $\bar{D}$ , it can be uniformly approximated in  $\bar{D}$  by polynomials. In this paper I shall construct a sequence of polynomials which converges uniformly to  $f(z)$  and, at the same time interpolates  $f(z)$  at a regular point system.

In 1942 ERDÖS proved the following result about real interpolation [3]. Let  $\{x_k^{(n)}\}$  be a regular point system in  $[-1, +1]$ . Then to every continuous function  $f(x)$  and  $\eta > 0$  there exists a sequence of polynomials  $p_n(x)$  such that, 1) the degree of  $p_n(x)$  is  $\leq n(1 + \eta)$ , 2)  $p_n(x_i^{(n)}) = f(x_i^{(n)})$ ,  $1 \leq i \leq n$ ,  $n=1, 2, \dots$ , 3)  $p_n(x) \rightarrow f(x)$  uniformly in  $[-1, +1]$ .

Adapting Erdős's proof, I shall prove the following

**THEOREM:** *Let  $D$  be a domain bounded by a closed Jordan curve  $\Gamma$  which satisfies Alper's smoothness condition (2.1). Let  $\{z_k^{(n)}\} \in \Gamma$  be a regular point system for  $\bar{D}$ . Then, to every function  $f(z)$  regular in  $D$  and continuous in  $\bar{D}$  and every  $\eta > 0$ , there exists a sequence of polynomials  $p_n(z)$  such that*

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<sup>1)</sup> The Fekete points  $\{w^{(n)}_k\}$  are defined by the property that for each  $n$ , they maximise the discriminant  $\prod_{\substack{1 \leq k, j \leq n \\ k \neq j}} |w_k - w_j|$ .

- 1) the degree of  $p_n(z) \leq n(1 + \eta)$
- 2)  $p_n(z_k^{(n)}) = f(z_k^{(n)})$ ,  $1 \leq k \leq n$ ,  $n = 1, 2, \dots$
- 3)  $p_n(z) \rightarrow f(z)$  uniformly in  $\bar{D}$
- 4)  $p_n = T_n(f)$  is a linear operator, given explicitly.

### 2. Preliminary results

Let  $\vartheta(s)$  denote the angle between the tangent to  $\Gamma$  and the positive real axis (as a function of the arc length parameter  $s$ ). Let  $\omega(h)$  denote the modulus of continuity of the function  $\vartheta(s)$ . In the papers [1,2] S. Y. ALPER introduced the class of domains whose boundary  $\Gamma$  satisfies the condition

$$\int_0^h \frac{\omega(x)}{x} |\log x| dx < +\infty \tag{2.1}$$

Assuming that condition (2.1) is satisfied, we are able to estimate certain fundamental polynomials.

Let

$$z = \psi(\zeta) = \beta\zeta + a_0 + \frac{a_1}{\zeta} + \dots$$

map  $|\zeta| > 1$  conformally onto the exterior of  $\Gamma$ . We can assume without loss of generality that  $\beta = 1$ . By a classical result  $\psi(\zeta)$  is continuous for  $|\zeta| \geq 1$ . Thus, the regular point system  $\{z_k^{(n)}\}$  can be written in the form:  $z_k^{(n)} = \psi(e^{i\vartheta_k^{(n)}})$ , where we can assume that  $\vartheta_1 < \vartheta_2 < \dots < \vartheta_n$ ,  $\vartheta_n - \vartheta_1 < 2\pi$ . In another paper [4, Theorem 3] POMMERENKE and I proved the following result:

LEMMA 2.1. *If  $\Gamma$  satisfies the condition (2.1) and  $\psi(e^{i\vartheta_k^{(n)}})$  is a regular point system, then:*

$$\vartheta_{k+v}^{(n)} - \vartheta_k^{(n)} > \frac{cv}{n}, \quad k = 1, 2, \dots, n, \quad v = 1, 2, \dots, n - 1 \quad (\vartheta_{n+j}^{(n)} = \vartheta_j^{(n)} + 2\pi) \tag{2.2}$$

where the constant  $c > 0$  does not depend on  $n$ ,  $k$  or  $v$ . To avoid an excess of indices, we shall now drop the index:  $(n)$ .

Let  $w_{k,j}^{(m)} = \psi(e^{i(\vartheta_k + 2\pi j/m)})$ ,  $1 \leq k \leq n$ ,  $0 \leq j < m$ ,  $w_{k,0}^{(m)} = z_k^{(n)}$  and let  $\tilde{l}_k^{(m)}(z)$  denote the fundamental polynomials:

$$\tilde{l}_k^{(m)}(z) = \prod_{j=1}^{m-1} \frac{z - w_{k,j}^{(m)}}{z_k - w_{k,j}^{(m)}}$$

In the above quoted paper [4, (4.3), (4.6)] it was shown, that if (2.1) is satisfied, one has the following estimates

LEMMA 2.2

$$(i) \quad |\tilde{i}_k^{(m)}(\psi(e^{i\vartheta}))| < B \tag{2.3}$$

$$(ii) \quad |\tilde{i}_k^{(m)}(\psi(e^{i\vartheta}))| < \frac{\pi B}{m|\vartheta - \vartheta_k|} \tag{2.4}$$

for  $-\pi \leq \vartheta - \vartheta_k \leq +\pi$ , where the constant  $B$  depends only on the domain  $D$ .

From lemmas 2.1 and 2.2 we deduce

LEMMA 2.3

$$\sum_{k=1}^n |\tilde{i}_k^{(m)}(z)|^2 \leq C_1 + C_2 \frac{n^2}{m^2}, \quad \text{for } z \in \bar{D} \tag{2.5}$$

where  $C_1$  and  $C_2$  does not depend on  $n$  or  $m$ .

*Proof:* Since the sum on the left hand side is subharmonic, it is sufficient to prove (2.5) for  $z \in \Gamma$ , i.e.  $z = \psi(e^{i\vartheta})$ . Without loss of generality we can also assume that  $\vartheta_n - 2\pi < \vartheta \leq \vartheta_1$ . Then, applying Lemma 2.2. and Lemma 2.1,

$$\begin{aligned} \sum_{k=1}^n |\tilde{i}_k^{(m)}(\psi(e^{i\vartheta}))|^2 &= |\tilde{i}_1^{(m)}(\psi(e^{i\vartheta}))|^2 \\ &+ \sum_{j=1}^{[(n+1)/2]-1} |\tilde{i}_{1+j}^{(m)}(\psi(e^{i\vartheta}))|^2 + \sum_{j=1}^{[n/2]-1} |\tilde{i}_{n-j}^{(m)}(\psi(e^{i\vartheta}))|^2 \\ &+ |\tilde{i}_n^{(m)}(\psi(e^{i\vartheta}))|^2 \\ &\leq B^2 + \frac{\pi^2 B^2}{m^2} \sum_{j=1}^{[(n+1)/2]-1} \frac{1}{(\vartheta - \vartheta_{1+j})^2} \\ &+ \frac{\pi^2 B^2}{m^2} \sum_{j=1}^{[n/2]-1} \frac{1}{(\vartheta - \vartheta_{n-j} + 2\pi)^2} + B^2 \\ &\leq 2B^2 + \frac{\pi^2 B^2}{m^2} \sum_{j=1}^{[(n+1)/2]-1} \frac{1}{(\vartheta_1 - \vartheta_{1+j})^2} \\ &+ \frac{\pi^2 B^2}{m^2} \sum_{j=1}^{[n/2]-1} \frac{1}{(\vartheta_n - \vartheta_{n-j})^2} \\ &\leq 2B^2 + \frac{2\pi^2 B^2 n^2}{c^2 m^2} \sum_{1 \leq j \leq n/2} \frac{1}{j^2} \\ &\leq 2B^2 + \frac{\pi^4 B^2 n^2}{c^2 m^2}. \end{aligned}$$

which proves (2.5).

*Remark.* It follows from (2.3), that if condition (2.1) is satisfied, the system:

$$z_k^{(n)} = \psi(e^{i(\alpha_n + 2\pi ik/n)}) \tag{2.6}$$

provides another example for *regular* point systems. Here the real numbers  $\alpha_1, \alpha_2, \alpha_3, \dots$  are completely arbitrary. We call the system (2.6) a system of *Fejèr-points*.

### 3. Proof of the theorem

Let  $\{q_n(z)\}$  be a sequence of polynomials with the property that

- (i)  $q_n(z)$  is of degree  $n$
- (ii)  $q_n(z) \rightarrow f(z)$  uniformly in  $\bar{D}$ .

By Walsh's classical result, such a sequence certainly exists. We will specify the choice of  $\{q_n(z)\}$  later. We have that

$$\varepsilon_n = \max_{z \in \bar{D}} |f(z) - q_n(z)| \rightarrow 0 \tag{3.1}$$

We now write  $m = [\frac{1}{2}\eta n]$ , and

$$p_{n-1}(z) = q_{n-1}(z) + \sum_{k=1}^n \{f(z_k) - q_{n-1}(z_k)\} l_k(z) [\tilde{l}_k^{(m)}(z)]^2 \tag{3.2}$$

Clearly:

$$p_{n-1}(z_v) = q_{n-1}(z_v) + f(z_v) - q_{n-1}(z_v) = f(z_v)$$

for  $v=1, 2, \dots, n$ . Further

$$|f(z) - p_{n-1}(z)| \leq |f(z) - q_{n-1}(z)| + \sum_{k=1}^n |f(z_k) - q_{n-1}(z_k)| |l_k(z)| |\tilde{l}_k^{(m)}(z)|^2.$$

Applying (3.1), (1.2), and (2.5):

$$\begin{aligned} |f(z) - p_{n-1}(z)| &\leq \varepsilon_n + M\varepsilon_n \sum_{k=1}^n |\tilde{l}_k^{(m)}(z)|^2 \\ &\leq M\varepsilon_n \left(1 + C_1 + C_2 \frac{n^2}{m^2}\right) \leq M\varepsilon_n \left(1 + C_1 + C_2 \frac{4}{\eta^2}\right) \text{ for } n \geq \frac{2}{\eta}. \end{aligned}$$

Hence

$$\max_{z \in \bar{D}} |f(z) - p_{n-1}(z)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus we have proved Theorem 3 with the exception of the last assertion. To complete the proof it only remains to specify the choice of the polynomials  $q_n(z)$ .

The polynomial

$$s_{n-1}(z) = \frac{1}{2\pi i} \int_{|t|=1} f(\psi(t)) \left[ \sum_{m=0}^{n-1} \left(1 - \frac{m}{n}\right) \frac{F_m(z)}{t^{m+1}} \right] dt$$

(where  $F_m(z)$  is the  $m$ -th Faber-polynomial of the domain  $D$ ) represents the arithmetic mean of the partial sums of the Faber-expansion of  $f(z)$ . It is known that for every  $f(z)$  regular in  $D$  and continuous in  $\bar{D}$ ,

$$s_n(z) \rightarrow f(z)$$

uniformly in  $\bar{D}$  (cf. [1]). Thus we can choose

$$q_n(z) = s_n(z).$$

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