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Canonical Vector Fields on Spheres

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§ 1. Introduction

We are interested in norm-preserving bilinear forms

$$M: R^r \otimes R^n \to R^n,$$

where $\otimes = \otimes_R$ and by norm preserving we mean $\|M(u \otimes v)\| = \|u\| \cdot \|v\|$. Such a form implies the existence of $r-1$ mutually orthonormal vector fields on S^{n-1} (see 1.2 below). Given n, the question of finding the largest r so that such a form exists was solved in 1923 by RADON [5], by HURWITZ [3], and again in 1942 by ECKMANN [2]. The methods of RADON and HURWITZ yield complicated iterative schemes for actually constructing the forms, which have recently been simplified by ADAMS, LAX, and PHILLIPS [1]. We now give a still simpler construction and prove certain relevant properties of the "canonical" vector fields thus obtained. In particular, they are closed under the intrinsic join operations of JAMES [4] (cf. Prop. 4.4).

Let M be a form as above and let $e_0, \ldots, e_{r-1}$ be an orthonormal basis for R^r. Then one obtains r orthogonal transformations $M_0, \ldots, M_{r-1} \in O(n)$ by defining

$$M_i(v) = M(e_i \otimes v), \quad 0 \leqslant i \leqslant r-1, \quad v \in R^n.$$

Conversely, M is defined by the M_i using the formula

$$M(u \otimes v) = \sum \alpha_i M_i(v), \quad \text{where} \quad u = \sum \alpha_i e_i \quad \text{and} \quad i = 0, \ldots, r-1.$$

1.1. THEOREM: *The following are equivalent*

A: *M is norm-preserving,*

B: $\langle M_i(v), M_j(v) \rangle = \delta_{ij} \|v\|^2 \ \forall\, 0 \leqslant i, j \leqslant r-1$ *and* $v \in R^n$,

C: $M_i \in O(n)$ and $M_i^t M_j + M_j^t M_i = 0$, $i \neq j$.

This theorem has been used in one form or another by most of the above authors, and its proof is omitted.

One can assume without loss of generality that $M_0 = \mathrm{id}$, by following M with M_0^{-1} if necessary. Then from (B) it follows that $\langle v, M_i(v) \rangle = 0$, $1 \leqslant i \leqslant r-1$, and hence if we restrict v to S^{n-1}, i.e. $\|v\| = 1$, we obtain

1.2. COROLLARY: *$M_1(v), \ldots, M_{r-1}(v)$ define a family of $r-1$ orthonormal vector* fields *on S^{n-1}.*

Furthermore, using (C) together with $M_0 = \mathrm{id}$ and $M_i^t M_i = 1$, we obtain

1.3. COROLLARY: $M_i + M_i^t = 0$, $M_i^2 = -1$, $M_i M_j + M_j M_i = 0$, $1 \leqslant i, j \leqslant r-1$.

1.4. DEFINITION: A norm preserving form $M: R^r \otimes R^n \to R^n$ is orthogonal to the identity if $\langle v, M(u \otimes v)\rangle = 0 \ \forall u \in R^r, v \in R^n$.

From the above remarks such a form is clearly equivalent to the existence of a norm preserving form $M_0' = \mathrm{id}$ and $M_i' = M_{i-1}$, $i \geqslant 1$. Furthermore, M then defines r orthonormal vector fields on S^{n-1} and M_i, M_j satisfy 1.3, $0 \leqslant i, j \leqslant r-1$.

We will use the notation $M_u = M(u \otimes -)$: $R^n \to R^n$, $u \in R^r$. Clearly $M_u / \|u\| \in O(n)$, and if M is orthogonal to id then M_u is antisymmetric. In all cases one has the following identity:

$$\langle M(u \otimes v_1), M(u \otimes v_2)\rangle = \langle M_u v_1, M_u v_2\rangle = \|u\|^2 \left\langle \frac{M_u}{\|u\|} v_1, \frac{M_u}{\|u\|} v_2 \right\rangle = \|u\|^2 \langle v_1, v_2\rangle .$$

§ 2. Tensor Products of Inner Product Spaces

Let V, W be inner product spaces over a field F. Then $V \otimes_F W$ is an inner product space, where

$$\langle v_1 \otimes w_1, v_2 \otimes w_2\rangle = \langle v_1, v_2\rangle \langle w_1, w_2\rangle .$$

In case $V = R^m$ and $W = R^n$, with their usual products, it is not hard to see that the resulting inner product on R^{mn} is also the usual one.

The following lemma will be exceedingly useful in the proof of Theorem 3.1.

2.1. ORTHOGONALITY LEMMA: *Let V, W be inner product spaces with commutative inner products and suppose $A: V \to V$ and $B: W \to W$ are endomorphisms such that*

(i) *A is orthogonal to* id_V, *that is $\langle v, Av\rangle = 0 \forall v \in V$, or B is orthogonal to* id_W

(ii) *A is symmetric and B antisymmetric, or vice-versa.*

Then the two endomorphisms $\varphi = A \otimes 1$ and $\psi = 1 \otimes B$ of $V \otimes W$ are orthogonal, that is $\langle \varphi a, \psi a\rangle = 0 \ \forall a \in V \otimes W$.

Proof: Let $a = \sum_i v_i \otimes w_i$. Then

$$\begin{aligned} \langle \varphi a, \psi a\rangle &= \langle \sum_i A v_i \otimes w_i, \sum_j v_j \otimes B w_j\rangle \\ &= \sum_{i,j} \langle A v_i, v_j\rangle \langle w_i, B w_j\rangle . \end{aligned}$$

Now (i) clearly implies that the terms where $i = j$ vanish. Then supposing $A^t = A$, $B^t = -B$, we have

$$\begin{aligned} \langle \varphi a, \psi a\rangle &= \sum_{i<j} (\langle A v_i, v_j\rangle \langle w_i, B w_j\rangle + \langle A v_j, v_i\rangle \langle w_j, B w_i\rangle) \\ &= \sum_{i<j} (\langle v_j, A v_i\rangle \langle B w_j, w_i\rangle + \langle v_j, A v_i\rangle \langle -B w_j, w_i\rangle) \\ &= 0 . \end{aligned}$$

REMARK: The representation $a = \sum_{i=1}^{t} v_i \otimes w_i$ is of course not unique. One can, however, always choose it so that $v_1, \ldots, v_t$ form a given basis of V, or similarly for the w_i (but not both).

§ 3. The Basic Construction

Let $C: R^8 \otimes R^8 \to R^8$ be the Cayley multiplication. Let $i: R^7 \to R^8$ be inclusion into the last seven co-ordinates, then $C \circ (i \otimes 1): R^7 \otimes R^8 \xrightarrow{C_1} R^8$ is a norm preserving multiplication orthogonal to the identity. Now define a form $N: R^7 \otimes R^{16} \to R^{16}$ by the composition

$$R^7 \otimes R^{16} \xrightarrow{\approx} R^7 \otimes R^8 \oplus R^7 \otimes R^8 \xrightarrow{C_1 \otimes (-C_1)} R^8 \oplus R^8 \xrightarrow{\approx} R^{16}.$$

Clearly N is norm preserving, orthogonal to id, and for $0 \leqslant i \leqslant 6$ each N_i is antisymmetric. Furthermore, N_i has the form

$$N_i = \begin{pmatrix} B_i & 0 \\ 0 & -B_i \end{pmatrix}, \qquad B_i \in O(8).$$

3.1. THEOREM: *Let $M: R^r \otimes R^n \to R^n$, n even, be a norm-preserving form such that*
(a) M *is orthogonal to* id
(b) $M_i U = -U M_i$, $0 \leqslant i \leqslant r-1$, *where* $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in O(n)$

Then the form $\bar{M}$ defined by the composition below is norm preserving and also satisfies (a), (b), (relative to r+8 and 16n):

$$\begin{tikzcd}
R^{r+8} \otimes R^{16n} \arrow[d, "\approx"] \arrow[dddr, "\bar{M}"] & \\
(R^r \oplus R^7 \oplus R^1) \otimes R^n \otimes R^{16} \arrow[d, "\approx"] & \\
(R^r \otimes R^n) \otimes R^{16} \oplus R^n \otimes (R^7 \otimes R^{16}) \oplus R^n \otimes R^{16} \arrow[d, "M \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}"', "+\, U \otimes N + 1 \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}"] & \\
R^n \otimes R^{16} \arrow[r, "\approx"] & R^{16n}
\end{tikzcd}$$

Proof: Condition (b) follows readily from the fact that $\bar{M}_i = \begin{pmatrix} A_i & 0 \\ 0 & -A_i \end{pmatrix}$, $0 \leqslant i < r+7$, while $\bar{M}_{r+7} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = T$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = V \in O(16)$.

From (b) it follows that $M_u U = -UM_u \ \forall u \in R^r$. Then UM_u is symmetric. Similarly, since $N_i = \begin{pmatrix} B_i & 0 \\ 0 & -B_i \end{pmatrix}$ satisfies (b), $0 \leqslant i \leqslant 6$, one sees that VN_u is antisymmetric $\forall u \in R^7$ and TN_u antisymmetric. Also, VT is symmetric.

Now, starting with $u \otimes v \in R^{r+8} \otimes R^{16n}$, let $u = u_1 \oplus u_2 \oplus u_3 \in R^r \oplus R^7 \oplus R^1$ and $v = \sum_i v_i' \otimes v_i'' \in R^n \otimes R^{16}$. Then $\bar{M}(u \otimes v) = a + b + c$, where

$$
\begin{aligned}
a &= \sum_j M(u_1 \otimes v_j') \otimes T v_j'', \\
b &= \sum_k U v_k' \otimes N(u_2 \otimes v_k''), \\
c &= u_3 \sum_l v_l' \otimes V v_l''.
\end{aligned}
$$

Toprove (a), we show $\langle v, a \rangle = \langle v, b \rangle = \langle v, c \rangle = 0$. $\langle v, a \rangle = \sum_{i,j} \langle v_i', M_{u_1}(v_j') \rangle \langle v_i'', T v_j'' \rangle$. Choosing $v_i'' = e_i$, the standard basis for R^{16}, $\langle v_i'', T v_j'' \rangle = \pm \delta_{ij}$ and $\langle v, a \rangle = \sum_i \pm \langle v_i', M_{u_1}(v_i') \rangle = 0$ since M is orthogonal to the identity. $\langle v, b \rangle = \sum_{i,1} \langle v_i', U v_k' \rangle \langle v_i'', N_{u_2} v_k'' \rangle = 0$ by the orthogonality lemma. $\langle v, c \rangle = u_3 \sum_{l,l} \langle v_i', v_l' \rangle \langle v_i'', V v_l'' \rangle = 0$ by choosing $\{v_i'\}$ orthonormal and noticing that V is orthogonal to id.

To show that $\bar{M}$ is norm preserving, we first prove that $\langle a, b \rangle = \langle a, c \rangle = \langle b, c \rangle = 0$.

$$
\begin{aligned}
\langle a, b \rangle &= \sum_{j,k} \langle M_{u_1}(v_j'), U v_k' \rangle \langle T v_j'', N_{u_2}(v_k'') \rangle \\
&= \sum_{j,k} \langle U M_{u_1}(v_j'), v_k' \rangle \langle v_j'', T N_{u_2}(v_k'') \rangle .
\end{aligned}
$$

Choosing $v_i'' = e_i$ as before, one has $\langle T v_i'', N_{u_2}(v_i'') \rangle = \pm \langle v_i'', N_{u_2}(v_i'') \rangle = 0$. Thus one need only consider the terms where $j \neq k$, which sum to zero since UM_{u_1} is symmetric and TN_{u_2} antisymmetric. The other two orthogonality relations are proved quite analogously, where in $\langle b, c \rangle$ one takes $\{v_i'\}$ to be the standard basis for R^n to insure that the (i, i) terms vanish. Thus

$$\|\bar{M}(u \otimes v)\|^2 = \|a\|^2 + \|b\|^2 + \|c\|^2 .$$

Choosing $v_i'' = e_i$, one easily sees that the individual terms in a, b, c are mutually orthogonal, being already orthogonal in the second factor. Then, since T, U, and V are all orthogonal transformations,

$$
\begin{aligned}
\|\bar{M}(u \otimes v)\|^2 &= \sum_i \|u_1\|^2 \|v_i'\|^2 \|v_i''\|^2 + \sum_i \|v_i'\|^2 \|u_2\|^2 \|v_i''\|^2 + u_3^2 \sum_i \|v_i'\|^2 \|v_i''\|^2 \\
&= (\|u_1\|^2 + \|u_2\|^2 + u_3^2) \sum_i \|v_i'\|^2 \|v_i''\|^2 \\
&= \|u\|^2 \|v\|^2 .
\end{aligned}
$$

3.2. Corollary: *If* $n=s\cdot 2^{4a+b}$, *s odd,* $0\leqslant b\leqslant 3$, *then* S^{n-1} *admits* $8a+2^b-1$ *orthonormal vector fields.*

Proof: If $n=s$ one has a trivial form $R^0\otimes R^s\xrightarrow{0}R^s$. Applying the theorem "$a$" times gives a norm preserving form orthogonal to the identity

$$R^{8a}\otimes R^{s\cdot 16^a}\to R^{s\cdot 16^a}$$

(the fact that n is odd on the first iteration causes no trouble since $r=0$ there). This is the case $b=0$. For $b=1$, 2, 3 one need only apply the theorem once more and observe that $\bar{M}(\mu R^{8a+2^b-1}\otimes\mu R^{s\cdot 16^a\cdot 2^b})\subset\mu R^{s\cdot 16^a 2^b}$, where μ denotes the generic inclusion of R^m into the first m co-ordinates of R^{m+k} for any m, k. This is so because $N(\mu R^{2^b-1}\otimes\mu R^{2^b})\subset\mu R^{2^b}$, $b=1$, 2, 3, corresponding to the complex numbers, quaternions, and Cayley numbers respectively.

Remark: $\varrho(n)=8a+2^b$ is called the Radon-Hurwitz function.

§ 4. Definition and Properties of Canonical Vector Fields

Let $R^\infty=\varinjlim R^m$. It is clear, using the definition of $\bar{M}$, that the following commutes:

$$\begin{tikzcd}
R^r\otimes R^n \arrow[r, "M"] \arrow[d, hook, "\mu\otimes\mu"] & R^m \arrow[d, hook, "\mu"] \\
R^{r+8}\otimes R^{16n} \arrow[r, "\bar{M}"] & R^{16n}
\end{tikzcd}$$

Starting with $R^0\otimes R^1\xrightarrow{0}R^1$, we now iterate Theorem 3.1 and pass to the limit, obtaining a norm preserving multiplication orthogonal to the identity

$$\mathbf{M}\colon R^\infty\otimes R^\infty\to R^\infty.$$

Let $\mu_i\colon R^m\hookrightarrow R^\infty$ be the inclusion of R^m into the i'th block of m co-ordinates, $0\leqslant i$. Thus, for $i\leqslant n$, one has a commutative diagram

$$\begin{tikzcd}
R^m \arrow[r, hook, "\mu_i"] \arrow[d, hook, "(\ )\otimes e_i"] & R^\infty \\
R^m\otimes R^n \arrow[r, hook, "\approx"] & R^{mn}. \arrow[u, hook, "\mu"]
\end{tikzcd}$$

Also, $\mu_0=\mu$.

The following theorem says that $\mathbf{M}$ in effect gives a maximal family of orthonormal vector fields on S^{n-1} for every n.

4.1. Theorem: *If* $r\leqslant\varrho(n)-1$ *then, for any* $i\geqslant 0$,

$$\mathbf{M}(\mu R^r\otimes\mu_i R^n)\subset\mu_i R^n.$$

Proof: This property will certainly hold for n if it is true for some divisor of n, the same r, and all i. Letting $n=s\cdot 2^{4a+b}$, s odd, $0\leqslant b\leqslant 3$, it will thus suffice to prove the theorem for 2^{4a+b}, since also $\varrho(2^{4a+b})=\varrho(n)$. In other words, we can assume without loss of generality that $n=2^{4a+b}$. Furthermore, if the result holds for $r=\varrho(n)-1$ it will certainly hold for smaller r, so we also take $r=\varrho(n)-1=8a+2^b-1$.

First consider $b=0$ and let $e_0, e_1, \ldots$ be the usual basis for R^∞. We shall prove that if the result holds for $0\leqslant i\leqslant 16^m-1$ then it also holds for $0\leqslant i\leqslant 16^{m+1}-1$, giving an inductive proof of the theorem for the case $b=0$ (clearly $m=0$ furnishes a base for the induction). Write $i=t\cdot 16^m+s$, where $0\leqslant s\leqslant 16^{m+a+1}$ and $0\leqslant t\leqslant 15$. The inclusion $R^n=R^{16^a}\xrightarrow{\mu_i}R^{16^{m+a+1}}$ corresponds to the composition

$$R^{16^a}\xrightarrow{\mu_s}R^{16^{m+a}}\xrightarrow{(\;)\otimes e_t}R^{16^{m+a}}\otimes R^{16}\xrightarrow{\approx}R^{16^{m+a+1}}.$$

Then in the passage from M to $\bar{M}$, i.e., from $R^{8(m+a)-1}\otimes R^{16^{m+a}}$ to $R^{8(m+a+1)-1}\otimes R^{16^{m+a+1}}$, we have a commutative diagram

$$\begin{tikzcd}
R^{8(m+a+1)-1}\otimes R^{16^{m+a+1}} \arrow[r, "\approx"] & R^{8(m+a)-1}\oplus R^{8})\otimes R^{16^{m+a}}\otimes R^{16} \arrow[d, "\approx"] \\
R^{8a-1}\otimes R^{16^a} \arrow[u, "\mu\otimes\mu_i"] \arrow[ur, "(\mu,0)\otimes\mu_s\otimes e_t"] \arrow[r, "(\mu\otimes\mu_s)\otimes e_t{,}\,0)"'] & (R^{8(m+a)-1}\otimes R^{16^{m+a}})\otimes R^{16}\oplus(R^{16^{m+a}}\otimes(R^{8}\otimes R^{16}))
\end{tikzcd}$$

Performing the multiplications and applying the inductive hypothesis, we find

$$\mathbf{M}(\mu R^{8a-1}\otimes\mu_i R^{16^a})\subset\mathbf{M}(\mu R^{8a}\otimes\mu_s R^{16^a})\otimes e_t\subset\mu_s R^{16^a}\otimes e_t,$$

and the latter corresponds to $\mu_i R^{16^a}$ under the isomorphism

$$R^{16^{(m+a)}}\otimes R^{16}\approx R^{16^{m+a+1}}.$$

This completes the proof for $b=0$. A similar method works for $b=1, 2, 3$. For example, if $b=2$, we use the existence of quaternions to establish the cases $0\leqslant i\leqslant 3$ (similar to the proof of Cor. 3.2) as base for the induction, then pass from $0\leqslant i\leqslant 4\cdot 16^m-1$ to $0\leqslant i\leqslant 4\cdot 16^{m+1}-1$

4.2. COROLLARY: *If* $r\leqslant\varrho(n)-1$ *then the following composition defines a norm preserving multiplication orthogonal to* id:

$$R^r\otimes R^n\xrightarrow{\mu\otimes\mu_i}R^\infty\otimes R^\infty\xrightarrow{\mathbf{M}}R^\infty\xrightarrow{\mu_i^{-1}}R^n.$$

Denoting this multiplication "$\mathbf{M}^n_{i,r}$", let us call the resultant r orthonormal vector fields on S^{n-1} "$f^n_{i,r}$". More precisely, $f^n_{i,r}: S^{n-1} \to V_{n,r+1}$ is the cross section of the fibration $V_{n,r+1} \to S^{n-1}$ such that

$$f^n_{i,r}(x) = \begin{pmatrix} x \\ x_0 \\ \vdots \\ x_{r-1} \end{pmatrix} \in V_{n,r+1}, \quad \text{where} \quad x_j = \mathbf{M}^n_{i,r}(e_j \otimes x).$$

These are our "canonical" vector fields.

4.3. Definition: Let $M: R^r \otimes R^m \to R^m$ and $N: R^r \otimes \mathrm{R}^n \to R^n$ be norm preserving forms orthogonal to the identity. Then their intrinsic join $M*N$ is the composition

$$R^r \otimes (R^m \oplus N) \xrightarrow{\approx} R^r \otimes R^m \oplus R^r \otimes R^n \xrightarrow{M \oplus N} R^m \oplus R^n \xrightarrow{\approx} R^{m+n}.$$

Clearly $M*N$ is also norm preserving and orthogonal to id. If $f: S^{m-1} \to V_{m,r+1}$ and $g: S^{n-1} \to V_{n,r+1}$ are the corresponding cross sections, then their intrinsic join $f*g$ is defined as the composition

$$S^{m+n-1} \underset{\varphi}{\xleftarrow{\cong}} S^{m-1} * S^{n-1} \xrightarrow{f*g} V_{m,r+1} * V_{n,r+1} \xrightarrow{\varphi} V_{m+n,r+1},$$

φ being the intrinsic join map of James [4]. One easily sees that $f*g$ corresponds to $M*N$, and it is then clear that the canonical vector fields can be joined together in many ways to give other canonical fields. A typical example is the formula

$$f^4_{0,3} * f^4_{1,3} = f^8_{0,3}.$$

More generally, one can easily establish the following.

4.4. Proposition: $f^n_{im,r} * f^n_{im+1,r} * \cdots * f^n_{im+m-1,r} = f^{mn}_{i,r}$

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