**Zeitschrift:** Commentarii Mathematici Helvetici

**Herausgeber:** Schweizerische Mathematische Gesellschaft

**Band:** 43 (1968)

**Artikel:** Zeta-Functions of Ideal Classes in Quadratic Fields and their Zeros on the Critical Line.

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**DOI:** https://doi.org/10.5169/seals-32904

# Zeta-Functions of Ideal Classes in Quadratic Fields and their Zeros on the Critical Line

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§ 1. If $K$ is a quadratic field, and $\mathfrak{C}$ is an ideal class in $K$, the Dedekind Zeta-function of the class $\mathfrak{C}$ in $K$ is defined by the Dirichlet series

$$\zeta_K(s, \mathfrak{C}) = \sum_{\mathfrak{a} \in \mathfrak{C}} \frac{1}{(N\mathfrak{a})^s}, \tag{1.1}$$

where $s$ is a complex variable, $s=\sigma+it$, $\sigma>1$; the sum extends over the non-zero integral ideals $\mathfrak{a}$ in $\mathfrak{C}$, and $N\mathfrak{a}$ is the norm of $\mathfrak{a}$. The function $\zeta(s, \mathfrak{C})$ satisfies a functional equation, the form of which depends on the nature of $K$. If $K$ is an imaginary quadratic field, say $K=Q(\sqrt{-d})$, $d>0$, then we have

$$\left(\frac{\sqrt{d}}{2\pi}\right)^s \Gamma(s)\,\zeta(s, \mathfrak{C}) = \left(\frac{\sqrt{d}}{2\pi}\right)^{1-s} \Gamma(1-s)\,\zeta(1-s, \mathfrak{C}). \tag{1.2}$$

If $K$ is a real quadratic field, say $K=Q(\sqrt{d})$, then the corresponding functional equation has a different gamma-factor, and is of the form

$$\left(\frac{\sqrt{d}}{\pi}\right)^s \Gamma^2\left(\frac{s}{2}\right) \zeta(s, \mathfrak{C}) = \left(\frac{\sqrt{d}}{\pi}\right)^{1-s} \Gamma^2\left(\frac{1-s}{2}\right) \zeta(1-s, \mathfrak{C}). \tag{1.3}$$

The equations (1.2) and (1.3) take this form since the field is quadratic, so that $\zeta(s, \mathfrak{C})=\zeta(s, \bar{\mathfrak{C}})$, where $\bar{\mathfrak{C}}$ is the class conjugate to $\mathfrak{C}$. It is known, after HECKE [1], that the Zeta-function of an ideal class in an imaginary quadratic field has an infinity of zeros on the critical line. It is not known, however, whether the corresponding result is true in the case of a real quadratic field. The Dirichlet series for $\zeta(s, \mathfrak{C})$, in both cases, can be written in the form

$$\zeta_K(s, \mathfrak{C}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \qquad \sigma > 1, \tag{1.4}$$

where

$$a_m = \sum_{\substack{\mathfrak{a} \in \mathfrak{C} \\ N\mathfrak{a}=m}} 1,$$

and it is known, after Dedekind, that

$$A(x) = \sum_{m \leqslant x} a_m \sim \kappa x, \qquad 0 < \kappa < \infty. \tag{1.5}$$

Our object is to give, in both cases, a sufficient condition, in terms of an estimate on $a_m$, for the existence of an infinity of zeros on the critical line, and to show that that estimate is actually true. Corresponding estimates exist for fields of degree $n>2$, and we postpone the more general problem to a later occasion.

What we actually require is that

$$\sum_{m \leqslant T} a_m e^{2\pi i m x} = o(T), \qquad \text{as} \qquad T \to \infty,$$

for any irrational $x$. This is obtained from Hermann Weyl's estimate of exponential sums. The connection between this estimate, and the existence of an infinity of zeros, on the critical line, for the corresponding Zeta-function, is established here by a combination of van der Corput's method [3, Ch. IV] for estimating exponential integrals, with the Hardy-Littlewood proof [3, p. 219] of Hardy's theorem establishing the existence of an infinity of zeros of the classical Riemann Zeta-function on the critical line. We prove the following simple results.

THEOREM 1. *For every irrational number $x$, we have the estimate*

$$\sum_{m \leqslant T} a_m e^{2\pi i m x} = o(T), \qquad \text{as} \qquad T \to \infty. \tag{1.6}$$

THEOREM 2. *The function $\zeta_K(\frac{1}{2}+it, \mathfrak{C})$ vanishes for an infinity of real values of $t$.*

§ 2. *Proof of Theorem* 1. We consider two cases, according as the given field $K$ is real or imaginary.

*Case (i)*. Let $K=Q(\sqrt{d}), d>0$. From the definition of $a_m$ it is seen that (refer, for example [5, p. 87]) if

$$S(T) = \sum_{m \leqslant T} a_m e^{2\pi i m x},$$

then

$$2S(T) = \sum e^{2\pi i |P(k,l)| (N(\mathfrak{b}))^{-1} x}. \tag{2.1}$$

Here $\mathfrak{b}$ is a non-zero integral ideal in the class $\mathfrak{C}^{-1}$ (where $\mathfrak{C}$ is the given ideal class) with a base $(a, b)$, and if $a', b'$ denote the conjugates of $a$ and $b$, then

$$P(k,l) = (ka + lb)(ka' + lb').$$

The summation in (2.1) is over integers $k$ and $l$, such that

$$|P(k,l)| \leqslant T(N\mathfrak{b}), \qquad 1 \leqslant \frac{|ka + lb|}{|ka' + lb'|} < \eta^2, \tag{2.2}$$

where $\eta$ is a fundamental unit. This set is the set of lattice points in the plane set $D_T$ defined by

$$|P(u, v)| \leqslant T(N\mathfrak{b}), \qquad 1 \leqslant \frac{|u\,a + v\,b|}{|u\,a' + v\,b'|} < \eta^2,$$

where $u$ and $v$ are real numbers. This plane set is clearly contained, for instance, in a square of side $c\sqrt{T}$, for a suitable constant $c>0$. And the intersection of any line in the $(u, v)$ plane with this set consists of at most two intervals. Thus

$$2|S(T)| \leqslant \sum_k \left| \sum_l e^{2\pi i |P(k,l)|(N\mathfrak{b})^{-1}x} \right|. \tag{2.3}$$

Now, for fixed $k$, $P(k, l)$ is a quadratic polynomial in $l$, with rational coefficients, and since $b\neq 0$, we see that $P(k, l)(N\mathfrak{b})^{-1}x$ is a quadratic polynomial with the leading coefficient irrational, and independent of $k$. Hence, by Weyl's inequality [4, § 3],

$$\sum_l e^{2\pi i |P(k,l)|(N\mathfrak{b})^{-1}x} = o(\sqrt{T}),$$

uniformly in $k$, where $l$ runs over any interval contained in $(0, c\sqrt{T})$. (The fact that we have $|P(k, l)|$ instead of $P$ makes no difference, since $P$ can have at most two changes of sign.) From (2.3) and the remark preceding it, it follows that $S(T)=o(T)$ for any irrational $x$.

*Case* (*ii*). Let $K=Q(\sqrt{-d})$. The argument here is similar. One has only to observe that [5, p. 88]

$$w\,S(T) = \sum e^{2\pi i |P(k,l)|(N\mathfrak{b})^{-1}x},$$

where $w$ is the number of roots of unity in $K$, and the summation is over the lattice-points $(k, l)$ in the domain defined by $|P(u, v)|<T(N\,\mathfrak{b})$, where $P(u, v)=(ua+vb)^2$.

§ 3. *Estimates of certain integrals.* For the proof of Theorem 2 we require a series of estimates of certain integrals. The method of obtaining them is by now classical, and was originated by van der Corput [3]. No attempt is made here to state the results with the fullest possible generality. The following is a variation of TITCHMARSH's exposition [3, Ch. IV].

Let $C^k[a, b]$ denote the class of real functions in $[a, b]$, which are $k$ times continuously differentiable.

LEMMA 1. *Let* $F\in C^1[a, b]$, *such that its first derivative* $F'$ *is monotonic, and* $|F'(x)|\geqslant m>0$ *throughout the interval* $a\leqslant x\leqslant b$. *Then*

$$\left| \int_a^b e^{iF(x)}\,dx \right| \leqslant \frac{4}{m}. \tag{3.1}$$

We can assume, by taking conjugates if necessary, that $F'$ is positive. Taking the real and imaginary parts of the integral separately, we see that

$$\int_a^b \cos\{F(x)\}\,dx = \int_a^b \frac{F'(x)\cos\{F(x)\}}{F'(x)}\,dx\,,$$

and an application of the second mean-value theorem gives

$$\left|\int_a^b \cos\{F(x)\}\,dx\right| \leqslant \frac{2}{m}\,.$$

Similarly also

$$\left|\int_a^b \sin\{F(x)\}\,dx\right| \leqslant \frac{2}{m}\,,$$

and hence the lemma.

LEMMA 2. *Let $F, G \in C^2[a, b]$,* and *$(F'/G)'$ have at most $p$ distinct zeros in $[a, b]$. Let*

$$\left|\frac{F'(x)}{G(x)}\right| \geqslant m > 0\,,$$

*throughout $[a, b]$. Then*

$$\left|\int_a^b G(x)\,e^{iF(x)}\,dx\right| \leqslant \frac{4(p+1)}{m}\,.$$

We divide the interval $[a, b]$ into at most $(p+1)$ intervals in each of which $G/F'$ is monotonic, and apply an argument similar to that of Lemma 1 in each of them.

LEMMA 3. *Let $F \in C^2[a, b]$, and $|F''(x)| \geqslant r > 0$, throughout $[a, b]$. Then*

$$\left|\int_a^b e^{iF(x)}\,dx\right| \leqslant \frac{8}{\sqrt{r}}\,.$$

*Proof.* We can assume, as before, that $F''(x) \geqslant r > 0$, which implies that $F'$ is monotone increasing, and therefore vanishes at most once in the interval $[a, b]$, say, at $c$. Let $\delta > 0$, and denote by $I$, $I_1$, $I_2$, $I_3$, the following intervals.

$$I = [a, b]\,, \qquad I_1 = \begin{cases} [a, c-\delta], & \text{if} \quad c-\delta > a\,, \\ \emptyset, & \text{if} \quad c-\delta \leqslant a\,. \end{cases}$$

$$I_2 = [c-\delta, c+\delta] \cap I \qquad I_3 = \begin{cases} [c+\delta, b]\,, & \text{if} \quad c+\delta < b\,, \\ \emptyset, & \text{if} \quad c+\delta \geqslant b\,. \end{cases} \tag{3.2}$$

We then have

$$\int_a^b e^{iF(x)}\,dx = \int_I e^{iF(x)}\,dx = \int_{I_1} + \int_{I_2} + \int_{I_3}.$$

It is immediate that $|\int_{I_2}| \leqslant 2\delta$, while in $I_3$, we have, if it is not empty,

$$F'(x) = \int_c^x F''(t)\,dt \geqslant r(x-c) \geqslant r\delta,$$

so that Lemma 1 gives $|\int_{I_3}| \leqslant 4/r\delta$, and similarly, if $I_1$ is not empty,

$$\left|\int_{I_1}\right| \leqslant 4/r\delta.$$

Hence

$$\left|\int_I\right| \leqslant 8/r\delta + 2\delta,$$

and if we choose $\delta = 2/\sqrt{r}$, we get what we want.

LEMMA 4. *Let $F \in C^2[a, b]$, and $|F''(x)| \geqslant r > 0$ throughout $[a, b]$. Let $G \in C^2[a, b]$; $|G(x)| \leqslant M$, for $a \leqslant x \leqslant b$; and $(F'/G)'$ have at most $p$ distinct zeros in $[a, b]$. Then*

$$\left|\int_a^b G(x)\,e^{iF(x)}\,dx\right| \leqslant \frac{8M(p+1)}{\sqrt{r}}.$$

The proof runs along the same lines as in the previous lemma, except that instead of Lemma 1, we now use Lemma 2.

LEMMA 5. *Let $F \in C^3[a, b]$, where $a > 0$. Assume that*

$$0 < \lambda_2 \leqslant |F''(x)| < A\lambda_2 \tag{3.3}$$

*and*

$$|F'''(x)| < A\lambda_3, \tag{3.4}$$

*throughout the interval $[a, b]$, where $A$ is some positive constant. Let $F'(c) = 0$, where $a \leqslant c \leqslant b$. Let $G(x)$ be a power of $x$, and $|G(x)| \leqslant M$ in $[a, b]$. Let $(F'/G)'$ have at most $d$ distinct zeros in $[a, b]$. We then have*

$$\begin{aligned}\int_a^b G(x)\,e^{iF(x)}\,dx = {} & G(c)\,(2\pi)^{1/2}\frac{e^{\pm i\pi/4 + iF(c)}}{|F''(c)|^{1/2}} + O\big((p+1)\,M\,\lambda_2^{-4/5}\lambda_3^{1/5}\big) + \\ & + O\left\{(\lambda_2\lambda_3)^{-2/5}\frac{M}{a}\right\} + O\left\{M\min\left(\frac{1}{|F'(a)|}, \lambda_2^{-1/2}\right)\right\} + \\ & + O\left\{M\min\left(\frac{1}{|F'(b)|}, \lambda_2^{-1/2}\right)\right\}.\end{aligned} \tag{3.5}$$

*Here the sign in* $e^{\pm i\pi/4}$ *is positive or negative according as* $F''$ *is positive or negative.*

*Proof.* Condition (3.3) implies that $F'$ is monotonic in $[a, b]$, so that it can vanish at only one point in $[a, b]$, namely the point $c$. As in (3.2), we define again the intervals $I_1$, $I_2$, $I_3$ in terms of a number $\delta>0$. We write

$$\int_a^b G(x)\, e^{iF(x)}\, dx = \int_{I_1} + \int_{I_2} + \int_{I_3},$$

and first consider $I_3$. If it is not empty, it can be divided into at most $(p+1)$ subintervals, the points of division being the zeros of $(F'/G)'$, on each of which the function $G/F'$ is monotonic. If, for instance, $[c_1, c_2]$ is one such interval in $[c+\delta, b]$, so that $c_2>c_1\geqslant c+\delta$, then, as in the proof of Lemma 1, we have

$$\int_{c_1}^{c_2} G(x)\, e^{iF(x)}\, dx = O\left\{M\left(\frac{1}{|F'(c_1)|} + \frac{1}{|F'(c_2)|}\right)\right\},$$

where

$$|F'(c_i)| = \left|\int_c^{c_i} F''(x)\, dx\right| \geqslant \delta\lambda_2, \qquad i = 1, 2,$$

so that

$$\int_{c_1}^{c_2} G(x)\, e^{iF(x)}\, dx = O\left(\frac{M}{\delta\lambda_2}\right),$$

and hence we obtain

$$\left|\int_{I_3}\right| = O\left(\frac{M(p+1)}{\delta\lambda_2}\right). \tag{3.6}$$

Similarly

$$\left|\int_{I_1}\right| = O\left(\frac{M(p+1)}{\delta\lambda_2}\right). \tag{3.7}$$

As for the integral over $I_2$, we observe that for $x\in I_2$ we have

$$G(x) = G(c) + O\big(|x-c| \sup_{I_2} |G'(x)|\big),$$

and since $G$ is a power of $x$, and $a>0$, this gives

$$G(x) = G(c) + O\left(\frac{\delta M}{a}\right),$$

so that

$$\int_{I_2} G(x)\, e^{iF(x)}\, dx = \int_{I_2} G(c)\, e^{iF(x)}\, dx + O\left(\frac{\delta^2 M}{a}\right), \tag{3.8}$$

Expanding $F$ in a neighbourhood of $c$, by Taylor's theorem, and noting that $F'(c)=0$, we get

$$\int_{I_2} G(c)\, e^{iF(x)}\, dx = G(c)\, e^{iF(c)} \int_{I_2} e^{(1/2)i(x-c)^2 F''(c)} [1 + O\{|x-c|^3 \lambda_3\}]\, dx$$

$$= G(c)\, e^{iF(c)} \int_{I_2} e^{(1/2)i(x-c)^2 F''(c)}\, dx + O(M\, \delta^4 \lambda_3). \tag{3.9}$$

The last integral can be written as

$$\int_{I_2} e^{(1/2)i(x-c)^2 F''(c)}\, dx = \int_{c-\delta}^{c+\delta} - E_1 - E_2, \tag{3.10}$$

where

$$E_1 = \begin{cases} 0, & \text{if} \quad c-\delta \geqslant a, \\ \int_{c-\delta}^{a}, & \text{if} \quad c-\delta < a, \end{cases}$$

and

$$E_2 = \begin{cases} 0, & \text{if} \quad c+\delta \leqslant b, \\ \int_{b}^{c+\delta}, & \text{if} \quad c+\delta > b. \end{cases}$$

By Lemma 1 we see that, if $c+\delta>b$,

$$\int_{b}^{c+\delta} e^{(1/2)i(x-c)^2 F''(c)}\, dx = O\left(\frac{1}{(b-c)\lambda_2}\right) = O\left(\frac{1}{|F'(b)|}\right).$$

On the other hand, by Lemma 3, the same integral is $O(1/\sqrt{\lambda_2})$, so that

$$E_2 = O\left\{\min\left(\frac{1}{|F'(b)|}, \frac{1}{\sqrt{\lambda_2}}\right)\right\}, \tag{3.11}$$

and similarly

$$E_1 = O\left\{\min\left(\frac{1}{|F'(a)|}, \frac{1}{\sqrt{\lambda_2}}\right)\right\}. \tag{3.12}$$

If we now assume that $F''(c)>0$, and write

$$u = \tfrac{1}{2}(x-c)^2 F''(c),$$

then

$$\int_{c-\delta}^{c+\delta} e^{(1/2)i(x-c)^2F''(c)}\,dx = \frac{\sqrt{2}}{\{F''(c)\}^{1/2}} \int_0^{(1/2)\delta^2 F''(c)} \frac{e^{iu}}{\sqrt{u}}\,du$$

$$= \frac{\sqrt{2}}{\{F''(c)\}^{1/2}}\left[\int_0^{\infty} \frac{e^{iu}}{\sqrt{u}}\,du + O\left(\frac{1}{\delta\sqrt{\lambda_2}}\right)\right] = \frac{(2\pi)^{1/2}\,e^{i\pi/4}}{\{F''(c)\}^{1/2}} + O\left(\frac{1}{\delta\lambda_2}\right).$$

If $F''(c)<0$, we can replace $F''(c)$ by its negative, and take conjugates. Thus, we have, in general

$$\int_{c-\delta}^{c+\delta} e^{(1/2)i(x-c)^2 F''(c)}\,dx = \frac{(2\pi)^{1/2}\,e^{\pm i\pi/4}}{|F''(c)|^{1/2}} + O\left(\frac{1}{\delta\lambda_2}\right). \tag{3.13}$$

If we now use the estimates given by (3.13), (3.12) and (3.11) in (3.10), combine it with (3.9), and revert to (3.8), we get

$$\begin{aligned}\int_{I_2} G(x)\,e^{iF(x)}\,dx &= \frac{G(c)(2\pi)^{1/2}\,e^{\pm i\pi/4}\,e^{iF(c)}}{|F''(c)|^{1/2}} + O\left(\frac{M}{\delta\lambda_2}\right) + \\ &+ O\left\{M\min\left(\frac{1}{|F'(a)|}, \frac{1}{\sqrt{\lambda_2}}\right)\right\} + O\left\{M\min\left(\frac{1}{|F'(b)|}, \frac{1}{\sqrt{\lambda_2}}\right)\right\} + \\ &+ O\left(\frac{\delta^2 M}{a}\right) + O(M\delta^4\lambda_3).\end{aligned} \tag{3.14}$$

If we choose $\delta=(\lambda_2\lambda_3)^{-1/5}$, and combine (3.14) with (3.7) and (3.6), we get the lemma.

§ 4. *Proof of Theorem 2.* If $K=Q(\sqrt{-d}), d>0$, the functional equation of $\zeta_K(s, \mathfrak{C})$ is given by (1.2), which can be written in the form $\zeta(s, \mathfrak{C})=\chi(s)\,\zeta(1-s, \mathfrak{C})$, where $\chi(s)=(\Gamma(1-s)/\Gamma(s))(\lambda/2\pi)^{1-2s}$, $\lambda=\sqrt{d}$. If $K=Q(\sqrt{d})$, the corresponding equation is (1.3), which can again be written as

$$\zeta(s, \mathfrak{C}) = \chi(s)\,\zeta(1-s, \mathfrak{C}),$$

where

$$\chi(s) = \frac{\Gamma^2\left(\frac{1-s}{2}\right)}{\Gamma^2(s/2)}\left(\frac{\lambda}{\pi}\right)^{1-2s}, \quad \lambda=\sqrt{d}.$$

*In either case*, we have $\chi(s)\,\chi(1-s)=1$, which implies that $\chi(\frac{1}{2}+it)\,\chi(\frac{1}{2}-it)=1$, from which it follows that $|\chi(\frac{1}{2}+it)|=1$, since $\chi(s)$ is real for real $s$. Let $\theta=\theta(t)=-\frac{1}{2}\arg\chi(\frac{1}{2}+it)$, so that $\chi(\frac{1}{2}+it)=e^{-2i\theta}$. Define

$$Z(t) = e^{i\theta}\zeta_K(\tfrac{1}{2}+it, \mathfrak{C}) = \{\chi(\tfrac{1}{2}+it)\}^{-1/2}\zeta_K(\tfrac{1}{2}+it, \mathfrak{C}). \tag{4.1}$$

Since $\Gamma(s)$ has no zeros, and only real poles, the function $\{\chi(s)\}^{-1}$ has a square root $\{\chi(s)\}^{-1/2}$ in the simply connected region $t > t_4$, if $t_4$ is large enough.

*In the case of equation* (1.2), we have

$$\{\chi(\tfrac{1}{2}+it)\}^{-1/2} = \left(\frac{\Gamma(\frac{1}{2}+it)}{\Gamma(\frac{1}{2}-it)}\right)^{1/2} \left(\frac{\lambda}{2\pi}\right)^{it},$$

and therefore

$$Z(t) = \left(\frac{\lambda}{2\pi}\right)^{it} \frac{\Gamma(\frac{1}{2}+it)}{|\Gamma(\frac{1}{2}-it)|} \zeta_K(\tfrac{1}{2}+it, \mathfrak{C}). \tag{4.2}$$

If we set

$$\xi(s) = \left(\frac{\lambda}{2\pi}\right)^{s} \Gamma(s)\zeta_K(s, \mathfrak{C}),$$

then the functional equation (1.2) takes the form $\xi(s) = \xi(1-s)$. Set $\psi(s) = s(s-1)\,\xi(s)$. Then $\psi(s) = \psi(1-s)$. If $s = \frac{1}{2}+iz$, where $z$ is complex, and

$$\Xi(z) = \psi(\tfrac{1}{2}+iz),$$

then

$$\Xi(z) = \psi(\tfrac{1}{2}+iz) = \psi(\tfrac{1}{2}-iz) = \Xi(-z).$$

Now $\xi(s)$ is real for real $s$, hence also $\psi(s)$. Therefore $\psi$ assumes conjugate values at conjugate points, hence $\psi(\frac{1}{2}+iz)$ is real for real $z$, and therefore $\Xi(t)$ is real for real $t$. Since

$$\Xi(t) = (\tfrac{1}{2}+it)(-\tfrac{1}{2}+it)\xi(\tfrac{1}{2}+it) = -(t^2+\tfrac{1}{4})\left(\frac{\lambda}{2\pi}\right)^{1/2+it} \Gamma(\tfrac{1}{2}+it)\zeta_K(\tfrac{1}{2}+it, \mathfrak{C}),$$

it follows, from (4.2), that

$$Z(t) = -\frac{\left(\frac{2\pi}{\lambda}\right)^{(1/2)}}{t^2+\frac{1}{4}} \cdot \frac{\Xi(t)}{|\Gamma(\frac{1}{2}-it)|},$$

and that

$$Z(t) \text{ is real for real } t, \tag{4.3}$$

and, because of (4.1), that

$$|Z(t)| = |\zeta_K(\tfrac{1}{2}+it, \mathfrak{C})|. \tag{4.4}$$

*The same is true also in the case of equation (1.3)*, corresponding to $K = Q(\sqrt{d})$. Now consider the integral

$$\int_{\mathscr{C}} \{\chi(s)\}^{-1/2} \zeta_K(s, \mathfrak{C})\, ds, \tag{4.5}$$

taken along the contour $\mathscr{C}$, which is a rectangle with sides $\sigma = \frac{1}{2}$, $\sigma = \frac{5}{4}$, $t = T$, and $t = 2T$, where $T > c > 0$. The integral vanishes by Cauchy's theorem. The contribution from that part of $\mathscr{C}$ which lies on $\sigma=\frac{1}{2}$ is

$$\int_{1/2+iT}^{1/2+2iT} \{\chi(s)\}^{-1/2}\zeta_K(s,\mathfrak{C})\,ds = i\int_T^{2T} Z(t)\,dt, \tag{4.6}$$

because of the definition of $Z(t)$ given in (4.1). To estimate the contributions from the other three sides of the rectangle, we use Stirling's formula, namely: in any fixed strip $-\infty<\alpha\leqslant\sigma\leqslant\beta<+\infty$, as $t\to+\infty$, we have

$$\Gamma(\sigma+it) = t^{\sigma+it-1/2}e^{-(1/2)\pi t-it+(1/2)i\pi(\sigma-1/2)}(2\pi)^{1/2}\{1+O(1/t)\}.$$

We then obtain, in the case of equation (1.2),

$$\{\chi(s)\}^{-1/2} = \left(\frac{\lambda}{2\pi}\right)^{\sigma-1/2+it}\left\{\frac{\Gamma(s)}{\Gamma(1-s)}\right\}^{1/2} = \left(\frac{\lambda}{2\pi}\right)^{\sigma-1/2+it} t^{\sigma+it-1/2}e^{-it}\{1+O(1/t)\}, \tag{4.7}$$

and in the case of equation (1.3), the only change is that there is an additional factor $e^{-i\pi/4}$. On the other hand, we have

$$\zeta_K(s,\mathfrak{C}) = O(t^{1-\sigma+\varepsilon}),\quad \tfrac{1}{2}\leqslant\sigma\leqslant 1,\quad \varepsilon>0, \tag{4.8}$$

as $t\to+\infty$. Hence

$$\{\chi(s)\}^{-1/2}\zeta_K(s,\mathfrak{C}) = \begin{cases} O(t^{\sigma-1/2}\cdot t^{1-\sigma+\varepsilon}) = O(t^{1/2+\varepsilon}), & \tfrac{1}{2}\leqslant\sigma\leqslant 1.\\ O(t^{\sigma-1/2+\varepsilon}) = O(t^{3/4+\varepsilon}), & 1<\sigma\leqslant\tfrac{5}{4}.\end{cases}$$

Hence the contributions from the integrals parallel to the real axis are $O(T^{3/4+\varepsilon})= o(T)$, if $\varepsilon<\frac{1}{4}$. The integral along the line $\sigma=\frac{5}{4}$ gives, because of (4.7),

$$c_1\cdot\int_T^{2T}\sum_{m=1}^{\infty}\frac{a_m}{m^{5/4+it}}\left(\frac{t\lambda}{2\pi}\right)^{3/4+it}e^{-it}\,dt, \tag{4.9}$$

where $c_1=1$ if $K=Q(\sqrt{-d})$, and $c_1=e^{-i\pi/4}$ if $K=Q(\sqrt{d})$, together with an $O$-term, which is

$$\int_T^{2T} O(t^{-1/4})\,dt = O(T^{3/4}).$$

If we put

$$F_m(t) = t\left(\log\frac{t\lambda}{2\pi} - \log m - 1\right),$$

then

$$F_m'(t) = \log\frac{t\lambda}{2\pi} - \log m,$$

so that $F_m'(t)=0$ for $t=2m\pi/\lambda$, while $F_m''(t)=1/t$, and $F_m'''(t)=-1/t^2$. We can then write the series in (4.9) as a constant multiple of

$$\sum_{m=1}^{\infty} \frac{a_m}{m^{5/4}} \int_T^{2T} t^{3/4} e^{iF_m(t)}\, dt = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3, \tag{4.10}$$

where $\sum_1$, $\sum_2$, $\sum_3$ are respecti vely the sums of the series extended over the range

$$1 \leqslant m \leqslant \tfrac{1}{2} T \cdot \frac{\lambda}{2\pi}, \quad \tfrac{1}{2} T \cdot \frac{\lambda}{2\pi} < m \leqslant 4T \cdot \frac{\lambda}{2\pi}, \quad m > \frac{2T\lambda}{\pi}.$$

In $\sum_1$ we have, for $T \leqslant t \leqslant 2T$, the inequality $F_m'(t) \geqslant \log 2$, and if we apply Lemma 2 (with $p=1$), we get

$$\sum\nolimits_1 = O\left(\sum\nolimits_1 \frac{a_m}{m^{5/4}} \cdot T^{3/4}\right) = O(T^{3/4}) = o(T). \tag{4.11}$$

In $\sum_3$, on the other hand, we have, for $T \leqslant t \leqslant 2T$, the inequality $F_m'(t) \leqslant -\log 2$, and if we apply Lemma 2, and use (1.5), we get

$$\sum\nolimits_3 = O\left(\sum\nolimits_3 \frac{a_m}{m^{5/4}} \cdot T^{3/4}\right) = O(T^{1/2}) = o(T). \tag{4.12}$$

Finally in $\sum_2$ we use Lemma 5 with the substitutions

$$F(t) = F_m(t), \quad c = \frac{2m\pi}{\lambda}, \quad |F''(c)|^{1/2} = \left(\frac{\lambda}{2m\pi}\right)^{1/2}, \quad G(c) = \left(\frac{2m\pi}{\lambda}\right)^{3/4}.$$

This can be done provided that a zero of $F'$ falls in the interval $[T, 2T]$, that is, if $T \leqslant 2m\pi/\lambda \leqslant 2T$. In that case we get, as the main term, a constant multiple of

$$\sum_{T \leqslant 2m\pi/\lambda \leqslant 2T} \frac{a_m}{m^{5/4}} \cdot e^{-2m\pi i/\lambda} \cdot m^{3/4+1/2} = \sum_{T \leqslant 2m\pi/\lambda \leqslant 2T} a_m e^{-2\pi i m x},$$

where $x$ is irrational, which is $o(T)$ by Theorem 1, together with an error term which, because $T=O(m)$, is of the order

$$O\left(\sum_{c_2 T \leqslant m \leqslant c_3 T} \frac{a_m}{m^{1/2}} \min\left(T^{1/2}, \frac{1}{\log(m/c_2 T)}\right)\right) + O\left(\sum_{c_2 T \leqslant m \leqslant c_3 T} \frac{a_m}{m^{1/2}} \min\right.$$
$$\left.\left(T^{1/2}, \frac{1}{\log(c_3 T/m)}\right)\right) + O\left(\sum_{c_2 T \leqslant m \leqslant c_3 T} \frac{a_m}{m^{5/4}} \cdot m^{3/4+2/5}\right) = O(T^{9/10}) = o(T).$$

If, however, we have $T/2 \leqslant 2m\pi/\lambda < T$, or $2T < 2m\pi/\lambda \leqslant 4T$, we write

$$\int_T^{2T} = \int_{T/2}^{2T} - \int_{T/2}^{T}, \quad \text{or} \quad \int_T^{2T} = \int_T^{4T} - \int_{2T}^{4T},$$

as the case may be, and apply Lemma 5 to each of the resulting integrals. The main term then cancels out, and the error terms in the integrals give again $O(m^{3/4+2/5})$, since $T = O(m)$, which leads to the same estimate as before. Hence

$$\textstyle\sum_2 = o(T). \tag{4.13}$$

If we combine (4.13), (4.12), and (4.11), with (4.10) and (4.9), we see that the contribution of the integral in (4.5) along the line $\sigma = \frac{5}{4}$ gives altogether $o(T)$. From (4.6) we conclude that

$$\int_T^{2T} Z(t) = o(T). \tag{4.14}$$

On the other hand, if $k$ is a positive integer such that $a_k \neq 0$, then we have

$$\int_T^{2T} |Z(t)|\, dt = \int_T^{2T} |\zeta_K(\tfrac{1}{2} + it, \mathfrak{C})|\, dt \geqslant \left| \int_T^{2T} \zeta_K(\tfrac{1}{2} + it, \mathfrak{C}) k^{it} dt \right|,$$

and

$$\begin{aligned} i\int_T^{2T} k^{it}\zeta_K(\tfrac{1}{2} + it, \mathfrak{C})\, dt &= \int_{1/2+iT}^{1/2+2iT} k^{s-1/2}\zeta_K(s, \mathfrak{C})\, ds \\ &= \int_{1/2+iT}^{2+iT} + \int_{2+iT}^{2+2iT} + \int_{2+2iT}^{1/2+2iT} = \int_{2+iT}^{2+2iT} + \int_{1/2}^{2} O(T^{1/2+\varepsilon})\, d\sigma \\ &= \left[ \frac{a_k}{k^{1/2}} s - \sum_{m \neq k} \frac{a_m}{k^{1/2}} \cdot \frac{1}{(m/k)^s \log m/k} \right]_{2+iT}^{2+2iT} + O(T^{1/2+\varepsilon}) \\ &= i\frac{a_k}{k^{1/2}} T + O(T^{1/2+\varepsilon}). \end{aligned}$$

Hence

$$\int_T^{2T} |Z(t)|\, dt > BT, \tag{4.15}$$

where $B$ is a positive constant. From (4.14), (4.15) and (4.3) it follows that $Z(t)$ cannot be ultimately of one sign. It then follows from (4.4) that $\zeta_K(\frac{1}{2} + it, \mathfrak{C})$ vanishes for an infinity of values of $t$, which proves Theorem 2.

§ 5. We conclude by remarking that the same argument gives the following

THEOREM 3. *If* $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \ldots, \mathfrak{C}_r$ *are ideal classes in a field* $K = Q(\sqrt{\pm d})$, *then the function*

$$\sum_{j=1}^{r} \alpha_j \zeta_K(s, \mathfrak{C}_j),$$

*where the coefficients* $\alpha_j$ *are real numbers, has infinitely many zeros on the line* $\sigma = \frac{1}{2}$.

We are thankful to Professor C. L. SIEGEL for his critical reading of the manuscript.

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Received June 24, 1967