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Galois cohomology of biquadratic extensions

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For a quadratic extension K/F of fields of characteristic different from 2, there is a well-known long exact sequence relating the Galois cohomology groups of (the absolute Galois group of) F and K with coefficients in $\mu_2 = \{\pm 1\}$; if $(a) \in H^1F$ is the element which corresponds to $K = F(\sqrt{a})$, the exact sequence is:

$$\cdots \longrightarrow H^{n-1}F \xrightarrow{(a)\cdot} H^nF \xrightarrow{\text{res}} H^nK \xrightarrow{\text{cor}} H^nF \xrightarrow{(a)\cdot} H^{n+1}F \longrightarrow \cdots \tag{1}$$

where res and cor are respectively the restriction and corestriction maps (see for instance [1, Cor. 4.6]). This exact sequence plays a crucial rôle in the investigation of quadratic forms under quadratic extensions: see [1], [2].

In the case of a biquadratic extension $M = F(\sqrt{a_1}, \sqrt{a_2})$, i.e. an elementary abelian Galois extension of degree 4, there is no such long exact sequence. However, the kernel of the restriction map $H^2F \to H^2M$ is known from Brauer group computations (see for instance [23, Cor. 2.8]); it consists of sums of symbols of the type $(a_1, x_1) + (a_2, x_2)$ where $x_1, x_2 \in F^\times$. Moreover, there is a "common slot lemma" which gives a criterion for such a sum to be zero: if $(a_1, x_1) + (a_2, x_2) = 0$, then there exists an element $y \in F^\times$ such that

$$(a_1, x_1) = (a_1, y) = (a_2, y) = (a_2, x_2).$$

This information can be encapsulated in the following exact sequence (see [23], [6, §3], [19]):

$$\bigoplus_{i=1}^{3} H^1L_i \xrightarrow{\beta_2} X \otimes_{\mathbb{F}_2} H^1F \xrightarrow{\gamma_2} H^2F \xrightarrow{\text{res}} H^2M$$

where L_1, L_2, L_3 are the quadratic extensions of F contained in M and X is the subgroup of H^1F generated by (a_1) and (a_2); γ_2 is the cup product and β_2 is defined below. (The corresponding sequence for triquadratic extensions is not exact in general: see [6, §5], [19, §5]).

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The aim of this paper is to extend and generalize this exact sequence, and to define a kind of "dual" sequence centered around the corestriction map. For each positive integer n, we define two 7-term zero-sequences S_n and S^n associated to a biquadratic extension M/F and show that exactness of both of these sequences depends only on the vanishing of two of their homology groups; we further show that these sequences are exact for $n \leq 2$.

To describe more explicitly the results of the paper, we fix the following notation: M/F is a biquadratic extension of fields of characteristic different from 2; $a_1, a_2, a_3 \in F^\times$ are such that $a_1 a_2 a_3 = 1$ and $M = F(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$. For $i = 1, 2, 3$ we let $L_i = F(\sqrt{a_i}) \subset M$ and we denote by σ_i the non-trivial element of the Galois group $G = \mathrm{Gal}(M/F)$ which leaves L_i elementwise invariant. We denote by $(a_i) \in H^1F$ the image of $a_i \in F^\times$ in $F^\times/F^{\times 2} = H^1F$ and we let $X = \{0, (a_1), (a_2), (a_3)\} \subset H^1F$. The group X is naturally identified to the dual of the Galois group G, by Kummer theory, since it is the kernel of the restriction map $H^1F \to H^1M$. For $i = 1, 2, 3$, we also choose a square root $\sqrt{a_i} \in L_i^\times$ in such a way that $\sqrt{a_1}\sqrt{a_2}\sqrt{a_3} = 1$, and we denote by $(\sqrt{a_i})$ its image in H^1L_i. The non-trivial element of $\mathrm{Gal}(L_i/F)$, which is the restriction of σ_j to L_i, for any $j \neq i$, is simply denoted by: $x \mapsto \bar{x}$. Finally, as a general rule, if K/E is an extension of fields, the image of the cohomology class $u \in H^nE$ under the restriction map $H^nE \to H^nK$ is simply denoted by u_K; if K/E is finite and separable, the corestriction map $H^nK \to H^nE$ is denoted by $N_{K/E}$ or $\mathrm{cor}_{K/E}$.

We define the following sequences, for $n \geq 0$:

$$S_n:\quad H^nM \oplus (H^nF)^3 \xrightarrow{\alpha_n} \bigoplus_{i=1}^{3} H^nL_i \xrightarrow{\beta_n} X \otimes H^nF \xrightarrow{\gamma_n} H^{n+1}F \xrightarrow{\text{res}} H^{n+1}M$$
$$\xrightarrow{\delta_n} \bigoplus_{i=1}^{3} H^{n+1}L_i \xrightarrow{\varepsilon_n} H^{n+1}M \oplus (X \otimes H^{n+1}F) \oplus H^{n+2}F$$

and

$$S^n:\quad H^{n+1}M \oplus (H^{n+1}F)^3 \xleftarrow{\alpha^n} \bigoplus_{i=1}^{3} H^{n+1}L_i \xleftarrow{\beta^n} G \otimes H^{n+1}F \xleftarrow{\gamma^n} H^nF \xleftarrow{\text{cor}} H^nM$$
$$\xleftarrow{\delta^n} \bigoplus_{i=1}^{3} H^nL_i \xleftarrow{\varepsilon^n} H^nM \oplus (G \otimes H^nF) \oplus H^{n-1}F$$

where the tensor products are over $\mathbb{F}_2$ (or $\mathbb{Z}$) and the maps are defined as follows:

- $\alpha_n(u, (v_i)_{1 \leq i \leq 3}) = (N_{M/L_i}(u) + (v_i)_{L_i})_{1 \leq i \leq 3}$
- $\beta_n(l_i)_{1 \leq i \leq 3} = \sum_{i=1}^{3} (a_i) \otimes N_{L_i/F}(l_i)$

- $\gamma_n\left(\sum_{i=1}^{3} (a_i)\otimes f_i\right) = \sum_{i=1}^{3} (a_i)\cdot f_i$
- $\delta_n(u) = (N_{M/L_i}(u))_{1\le i\le 3}$
- $\varepsilon_n(l_i)_{1\le i\le 3} = \left(\sum_{i=1}^{3} (\bar{l}_i)_M, \beta_{n+1}(l_i)_{1\le i\le 3}, \sum_{i=1}^{3} N_{L_i/F}((\sqrt{a_i})\cdot l_i)\right)$

and, denoting by $\langle\,,\rangle : X\times G \to \mathbb{F}_2$ the canonical pairing,

- $\alpha^n(l_i)_{1\le i\le 3} = \left(\sum_{i=1}^{3} (l_i)_M, (N_{L_i/F}(l_i))_{1\le i\le 3}\right)$
- $\beta^n\left(\sum_{i=1}^{3} \sigma_i\otimes u_i\right) = \left(\left(\sum_{j=1}^{3} \langle (a_i), \sigma_j\rangle u_j\right)_{L_i}\right)_{1\le i\le 3}$, so that

  $\beta^n(\sigma_1\otimes u_1 + \sigma_2\otimes u_2 + \sigma_3\otimes u_3) = ((u_2+u_3)_{L_1}, (u_1+u_3)_{L_2}, (u_1+u_2)_{L_3})$
- $\gamma^n(u) = \sum_{i=1}^{3} \sigma_i\otimes (a_i)\cdot u$
- $\delta^n(l_i)_{1\le i\le 3} = \sum_{i=1}^{3} (l_i)_M$
- $\varepsilon^n\left(u, \sum_{i=1}^{3} \sigma_i\otimes v_i, t\right) = \overline{(N_{M/L_i}(u)} + ((\sqrt{a_i})t_{L_i}))_{1\le i\le 3} + \beta^{n-1}\left(\sum_{i=1}^{3} \sigma_i\otimes v_i\right).$

(In sequence S^0, we set $H^{-1}F = 0$).

The fact that S_n and S^n are zero-sequences is easily checked. (To see that $\varepsilon_n\cdot\delta_n = 0$, observe that

$$\sum_{i=1}^{3} N_{L_i/F}((\sqrt{a_i})\cdot N_{M/L_i}(u)) = N_{M/F}\left(\sum_{i=1}^{3} (\sqrt{a_i})_M\cdot u\right) = 0,$$

since the square roots have been chosen in such a way that $(\sqrt{a_1})_M + (\sqrt{a_2})_M + (\sqrt{a_3})_M = 0$). Therefore, we may define homology groups:

$\mathscr{H}_n(1) = \operatorname{Ker}\beta_n/\operatorname{Im}\alpha_n$	$\mathscr{H}^n(1) = \operatorname{Ker}\alpha^n/\operatorname{Im}\beta^n$
$\mathscr{H}_n(2) = \operatorname{Ker}\gamma_n/\operatorname{Im}\beta_n$	$\mathscr{H}^n(2) = \operatorname{Ker}\beta^n/\operatorname{Im}\gamma^n$
$\mathscr{H}_n(3) = \operatorname{Ker}\operatorname{res}/\operatorname{Im}\gamma_n$	$\mathscr{H}^n(3) = \operatorname{Ker}\gamma^n/\operatorname{Im}\operatorname{cor}$
$\mathscr{H}_n(4) = \operatorname{Ker}\delta_n/\operatorname{Im}\operatorname{res}$	$\mathscr{H}^n(4) = \operatorname{Ker}\operatorname{cor}/\operatorname{Im}\delta^n$
$\mathscr{H}_n(5) = \operatorname{Ker}\varepsilon_n/\operatorname{Im}\delta_n$	$\mathscr{H}^n(5) = \operatorname{Ker}\delta^n/\operatorname{Im}\varepsilon^n.$

The results we prove are the following:

THEOREM A. *For all $n \geq 0$, there are natural isomorphisms*: $\mathscr{H}_n(2) \simeq \mathscr{H}^n(2) \simeq \mathscr{H}_n(4) \simeq \mathscr{H}^n(4)$ *and* $\mathscr{H}_n(3) \simeq \mathscr{H}^n(3)$. *Moreover, if* $\mathscr{H}_n(3) = \mathscr{H}^n(3) = 0$, *then* $\mathscr{H}_n(1) = \mathscr{H}^n(1) = \mathscr{H}_n(5) = \mathscr{H}^n(5) = 0$.

THEOREM B. *The sequences S_n and S^n are exact for $n \leq 2$. Moreover,*

$$\mathscr{H}_3(1) = \mathscr{H}^3(1) = \mathscr{H}_3(3) = \mathscr{H}^3(3) = \mathscr{H}_3(5) = \mathscr{H}^3(5) = 0.$$

In the case where M is not a field but only an étale algebra Galois over the field F with Galois group G elementary abelian of order 4, the sequences S_n and S^n are exact for all n, provided $(\sqrt{a_i}) \in H^1 L_i$ is suitably chosen when L_i is split: if $L_i \simeq F \times F$, one sets $(\sqrt{a_i}) = ((-1), 0)$ or $(0, (-1)) \in H^1 F \times H^1 F$, adjusting the various choices in such a way that $(\sqrt{a_1})_M + (\sqrt{a_2})_M + (\sqrt{a_3})_M = 0$. Exactness of S_n and S^n in the completely split case (i.e. when $M \simeq F^4$) is easily checked by elementary computations; in the case where $M \simeq L \times L$ for some field L quadratic over F, exactness of S_n and S^n follows from the exact sequence (1) for quadratic extensions.

1. Proof of Theorem A

Our method of proof uses the following easy observation: consider two intersecting exact sequences:

$$\begin{array}{ccccccccc}
 & & & & C & & & & \\
 & & & & \downarrow{\scriptstyle \iota} & & & & \\
 & & & & D & & & & \\
 & & & & \downarrow{\scriptstyle d} & & & & \\
\Gamma & \xrightarrow{\gamma} & \Delta & \xrightarrow{\delta} & E & \xrightarrow{\varepsilon} & Z & \xrightarrow{\zeta} & H \\
 & & & & \downarrow{\scriptstyle e} & & & & \\
 & & & & F & & & & \\
 & & & & \downarrow{\scriptstyle f} & & & & \\
 & & & & G & & & &
\end{array}$$

We then get two zero-sequences:

$$C \xrightarrow{\iota} D \xrightarrow{\varepsilon d} Z \xrightarrow{\zeta} H \tag{2}$$

and

$$\Gamma \xrightarrow{\gamma} \Delta \xrightarrow{\delta} F \xrightarrow{f} G. \tag{3}$$

We denote the homology groups of (2) at D and Z by $\mathscr{H}(D)$ and $\mathscr{H}(Z)$ respectively and the homology groups of (3) at Δ and F by $\mathscr{H}(\Delta)$ and $\mathscr{H}(F)$ respectively.

LEMMA (of the 700th[1]). *There are natural exact sequences*:

$$0 \longrightarrow \mathscr{H}(D) \xrightarrow{d} E \xrightarrow{e \oplus \varepsilon} F \oplus Z$$

$$0 \longrightarrow \mathscr{H}(\Delta) \xrightarrow{\delta} E \xrightarrow{e \oplus \varepsilon} F \oplus Z$$

$$D \oplus \Delta \xrightarrow{d+\delta} E \xrightarrow{\varepsilon} \mathscr{H}(Z) \longrightarrow 0$$

$$D \oplus \Delta \xrightarrow{d+\delta} E \xrightarrow{e} \mathscr{H}(F) \longrightarrow 0.$$

In particular, there are natural isomorphisms: $\mathscr{H}(D) \simeq \mathscr{H}(\Delta)$ *and* $\mathscr{H}(Z) \simeq \mathscr{H}(F)$.

The proof is a straightforward verification, which is left to the reader.

Henceforth, we use the same notation as in the introduction; for simplicity, we denote by Λ the field $\mathbb{F}_2$ with two elements. If Γ denotes the absolute Galois group of F, we then have $H^nF = H^n(\Gamma, \Lambda)$; moreover, Shapiro's lemma shows that the cohomology of Γ with coefficients in the group algebra $\Lambda[G]$ is canonically isomorphic to the cohomology of M:

$$H^nM = H^n(\Gamma, \Lambda[G]).$$

For $i = 1, 2, 3$, let $G_i = \mathrm{Gal}\,(L_i/F)$; we denote by e_i and $\bar{e}_i$ respectively the trivial and the non-trivial element of G_i viewed in $\oplus_{i=1}^3 \Lambda[G_i]$. Thus, $(e_1, \bar{e}_1, e_2, \bar{e}_2, e_3, \bar{e}_3)$ is a basis of $\oplus_{i=1}^3 \Lambda[G_i]$ as a vector space over Λ. There is a fundamental exact sequence of Γ-modules:

$$0 \longrightarrow \Lambda \xrightarrow{\mathrm{res}} \Lambda[G] \xrightarrow{f} \bigoplus_{i=1}^{3} \Lambda[G_i] \xrightarrow{g} \Lambda[G] \xrightarrow{\mathrm{cor}} \Lambda \longrightarrow 0 \tag{4}$$

[1] We thank M.-A. Knus for suggesting this name to us.

where the maps are defined as follows:

- res (1) $= 1 + \sigma_1 + \sigma_2 + \sigma_3$. This map induces the restriction map in cohomology.
- $f(1) = e_1 + e_2 + e_3$; the fact that f is a Γ-module homomorphism then implies: $f(\sigma_1) = e_1 + \bar{e}_2 + \bar{e}_3$, $f(\sigma_2) = \bar{e}_1 + e_2 + \bar{e}_3$ and $f(\sigma_3) = \bar{e}_1 + \bar{e}_2 + e_3$. This map induces the direct sum of corestriction maps in cohomology.
- $g(\bar{e}_i) = 1 + \sigma_i$ for $i = 1, 2, 3$; the fact that g is a Γ-module homomorphism then implies $g(e_i) = \sigma_j + \sigma_k$ where $\{i, j, k\} = \{1, 2, 3\}$. This map induces in cohomology the sum of restrictions of conjugates: $g : \oplus_{i=1}^{3} H^n L_i \to H^n M$ maps (l_1, l_2, l_3) to $(\bar{l}_1)_M + (\bar{l}_2)_M + (\bar{l}_3)_M$.
- cor is the augmentation map: cor (1) = cor (σ_1) = cor (σ_2) = cor (σ_3) = 1. This map induces the corestriction map in cohomology.

The exactness of the sequence (4) is easily checked by a straightforward computation.

The Λ-vector space $\Lambda[G]$ can be identified with its dual by using the canonical symmetric bilinear form defined by the trace map $t : \Lambda[G] \to \Lambda$ which carries every element in $\Lambda[G]$ to the coefficient of 1. Similarly, $(\oplus_{i=1}^{3} \Lambda[G_i])^* = \oplus_{i=1}^{3} \Lambda[G_i]$. The transpose of res : $\Lambda \to \Lambda[G]$ is then cor : $\Lambda[G] \to \Lambda$, and the transpose of the fundamental sequence (4) is:

$$0 \longrightarrow \Lambda \xrightarrow{\text{res}} \Lambda[G] \xrightarrow{g^*} \bigoplus_{i=1}^{3} \Lambda[G_i] \xrightarrow{f^*} \Lambda[G] \xrightarrow{\text{cor}} \Lambda \longrightarrow 0 \qquad (5)$$

where $g^*(1) = \bar{e}_1 + \bar{e}_2 + \bar{e}_3$, $g^*(\sigma_1) = \bar{e}_1 + e_2 + e_3$, $g^*(\sigma_2) = e_1 + \bar{e}_2 + e_3$, $g^*(\sigma_3) = e_1 + e_2 + \bar{e}_3$ and $f^*(e_i) = 1 + \sigma_i$ for all i; $f^*(\bar{e}_i) = \sigma_j + \sigma_k$ where $\{i, j, k\} = \{1, 2, 3\}$. The map g^* induces in cohomology the direct sum of corestriction of conjugates:

$$g^* : H^n M \longrightarrow \bigoplus_{i=1}^{3} H^n L_i \text{ maps } u \text{ to } (\overline{N_{M/L_i}(u)})_{1 \le i \le 3}$$

and f^* induces the sum of restriction maps in cohomology.

Let N denote the cokernel of the map res:

$$N = \text{coker}\,(\text{res} : \Lambda \to \Lambda[G]);$$

by duality, we get

$$N^* = \ker\,(\text{cor} : \Lambda[G] \to \Lambda),$$

and the exact sequences (4) and (5) yield four short exact sequences:

$$0 \longrightarrow \Lambda \xrightarrow{\text{res}} \Lambda[G] \xrightarrow{\pi} N \longrightarrow 0 \tag{6}$$

$$0 \longrightarrow N \xrightarrow{f} \bigoplus_{i=1}^{3} \Lambda[G_i] \xrightarrow{g} N^* \longrightarrow 0 \tag{7}$$

$$0 \longrightarrow N \xrightarrow{g^*} \bigoplus_{i=1}^{3} \Lambda[G_i] \xrightarrow{f^*} N^* \longrightarrow 0 \tag{8}$$

$$0 \longrightarrow N^* \xrightarrow{i} \Lambda[G] \xrightarrow{\text{cor}} \Lambda \longrightarrow 0, \tag{9}$$

where π and i are the canonical maps.

We further define an exact sequence

$$0 \longrightarrow G \xrightarrow{h} N \xrightarrow{\text{cor}} \Lambda \longrightarrow 0 \tag{10}$$

and its dual: .

$$0 \longrightarrow \Lambda \xrightarrow{\text{res}} N^* \xrightarrow{h^*} X \longrightarrow 0 \tag{11}$$

where h is defined by: $h(\sigma_i) = 1 + \sigma_i + \text{res}(\Lambda)$ for $i = 1, 2, 3$, and the maps cor and res are induced by the corresponding maps in (4) (or (5)). The sequences (10) and (11) are exact sequences of Γ-modules, provided G and X are endowed with the trivial action of Γ; we therefore have $H^n(\Gamma, G) = G \otimes_\Lambda H^n(\Gamma, \Lambda) = G \otimes H^nF$ and, similarly, $H^n(\Gamma, X) = X \otimes H^nF$.

Observe that, even though the sequences (7) and (8) are not identical, the corresponding connecting maps $\partial : H^n(\Gamma, N^*) \longrightarrow H^{n+1}(\Gamma, N)$ are the same, since there is a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{f} & \bigoplus_{i=1}^{3} \Lambda[G_i] & \xrightarrow{g} & N^* & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \xrightarrow{g^*} & \bigoplus_{i=1}^{3} \Lambda[G_i] & \xrightarrow{f^*} & N^* & \longrightarrow & 0
\end{array}$$

where the central vertical map interchanges e_i and $\bar{e}_i$ for all i.

LEMMA 1. *The sequence*:

$$H^nF \xrightarrow{\text{res}} H^n(\Gamma, N^*) \xrightarrow{\partial} H^{n+1}(\Gamma, N) \xrightarrow{\text{cor}} H^{n+1}F$$

is a zero-sequence for all $n \geq 0$.

Proof. There is a commutative diagram of Γ-modules:

$$\begin{array}{ccccccccc}
 & & & & \Lambda & = & \Lambda & & \\
 & & & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \mathrm{res}} & & \\
0 & \longrightarrow & N & \xrightarrow{f} & \bigoplus_{i=1}^{3}\Lambda[G_i] & \xrightarrow{g} & N^* & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \mathrm{cor}} & & \downarrow{\scriptstyle l} & & & & \\
 & & \Lambda & = & \Lambda & & & &
\end{array}$$

where $k(1) = e_1 + \bar{e}_1 + e_2 + \bar{e}_2 + e_3 + \bar{e}_3$ (i.e. k induces the direct sum of restriction maps) and $l(e_i) = l(\bar{e}_i) = 1$ for $i = 1, 2, 3$ (i.e. l induces the sum of corestriction maps). The lemma readily follows from the induced commutative diagram in cohomology:

$$\begin{array}{ccccccc}
H^nF & = & H^nF & & & & \\
\downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \mathrm{res}} & & & & \\
\bigoplus_{i=1}^{3} H^nL_i & \xrightarrow[g]{} & H^n(\Gamma, N^*) & \xrightarrow{\partial} & H^{n+1}(\Gamma, N) & \xrightarrow{f} & \bigoplus_{i=1}^{3} H^{n+1}L_i\\
 & & & & \downarrow{\scriptstyle \mathrm{cor}} & & \downarrow{\scriptstyle l}\\
 & & & & H^{n+1}F & = & H^{n+1}F
\end{array}$$

□

COROLLARY 1. *There are natural isomorphisms*:

$$\mathscr{H}_n(2) \simeq \mathrm{Im}\,(\partial : H^n(\Gamma, N^*) \longrightarrow H^{n+1}(\Gamma, N)) \simeq \mathscr{H}^n(2).$$

Proof. The cohomology sequences associated to (7) and (11) intersect at $H^n(\Gamma, N^*)$:

$$\begin{array}{ccccccc}
 & & \bigoplus_{i=1}^{3} H^nL_i & & & & \\
 & & \downarrow{\scriptstyle g} & & & & \\
H^nF & \xrightarrow{\mathrm{res}} & H^n(\Gamma, N^*) & \xrightarrow{h^*} & X \otimes H^nF & \longrightarrow & H^{n+1}F\\
 & & \downarrow{\scriptstyle \partial} & & & & \\
 & & H^{n+1}(\Gamma, N) & & & & \\
 & & \downarrow{\scriptstyle f} & & & & \\
 & & \bigoplus_{i=1}^{3} H^{n+1}L_i & & & &
\end{array}$$

By the lemma of the 700th, the homology of

$$H^nF \xrightarrow{\partial\cdot\mathrm{res}} H^{n+1}(\Gamma, N) \xrightarrow{f} \bigoplus_{i=1}^{3} H^{n+1}L_i,$$

which is Im (∂) since the preceding lemma shows $\partial \cdot \mathrm{res} = 0$, is naturally isomorphic to the homology of

$$\bigoplus_{i=1}^{3} H^nL_i \xrightarrow{h^*} X \otimes H^nF \xrightarrow{g} H^{n+1}F.$$

A straightforward computation shows that $h^* \cdot g = \beta_n$ and that γ_n is the connecting homomorphism in the cohomology sequence associated to (11). This proves the first part of the corollary. To prove the second part, we let the cohomology sequences associated to (7) and (10) intersect at $H^{n+1}(\Gamma, N)$ and argue as above, using the fact that cor $\cdot\, \partial = 0$. □

For all $n \geq 0$, let $\partial_r : H^n(\Gamma, N) \to H^{n+1}F$ and $\partial_c : H^nF \to H^{n+1}(\Gamma, N^*)$ denote the connecting homomorphisms in the cohomology exact sequences associated to (6) and (9) respectively.

LEMMA 2. *The sequence*:

$$H^{n-1}F \xrightarrow{\partial_c} H^n(\Gamma, N^*) \xrightarrow{\partial} H^{n+1}(\Gamma, N) \xrightarrow{\partial_r} H^{n+2}F$$

is a zero-sequence for all $n \geq 0$. (*We let* $H^{n-1}F = 0$ *if* $n = 0$).

Proof. Since $(\sqrt{a_i}) \in H^1L_i = H^1(\Gamma, \Lambda[G_i])$ for $i = 1, 2, 3$, we may consider

$$((\sqrt{a_1}), (\sqrt{a_2}), (\sqrt{a_3})) \in H^1\left(\Gamma, \bigoplus_{i=1}^{3} \Lambda[G_i]\right);$$

let

$$0 \longrightarrow \bigoplus_{i=1}^{3} \Lambda[G_i] \longrightarrow U \longrightarrow \Lambda \longrightarrow 0$$

be a corresponding extension of Γ-modules (using the natural isomorphism between $H^1(\Gamma, \oplus_{i=1}^3 \Lambda[G_i])$ and $\mathrm{Ext}^1_{\Lambda[\Gamma]}(\Lambda, \oplus_{i=1}^3 \Lambda[G_i])$). More explicitly, a base of U as a Λ-vector space is given by $e_1, \bar{e}_1, e_2, \bar{e}_2, e_3, \bar{e}_3$ and an extra element u on which Γ acts by: $\tau(u) = u + s(\tau)$, where s is a 1-cocycle whose cohomology class is $((\sqrt{a_1}), (\sqrt{a_2}), (\sqrt{a_3}))$. Since $(\sqrt{a_1})_M + (\sqrt{a_2})_M + (\sqrt{a_3})_M = 0$, the image of this cohomology class in $H^1(\Gamma, \Lambda[G]) = H^1M$ under f^* is trivial; therefore, there is a commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i=1}^{3} \Lambda[G_i] & \longrightarrow & U & \longrightarrow & \Lambda & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f^*} & & \downarrow & & \| & & \\
0 & \longrightarrow & N^* & \xrightarrow{i} & \Lambda[G] & \xrightarrow{\text{cor}} & \Lambda & \longrightarrow & 0
\end{array}
$$

where the central vertical map carries u to an element $\lambda \in \Lambda[G]$ such that $f^*(s(\tau)) = \tau(\lambda) - \lambda$ for all $\tau \in \Gamma$. This diagram yields a commutative diagram in cohomology:

$$
\begin{array}{ccc}
H^nF & \longrightarrow & \bigoplus_{i=1}^{3} H^{n+1}L_i \\
\| & & \downarrow{\scriptstyle f^*} \\
H^nF & \xrightarrow{\partial_c} & H^{n+1}(\Gamma, N^*) \\
 & & \downarrow{\scriptstyle \partial} \\
 & & H^{n+2}(\Gamma, N)
\end{array}
$$

from which it follows that $\partial \cdot \partial_c = 0$. To prove $\partial_r \cdot \partial = 0$, we consider the dual of the first diagram of this proof:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda & \xrightarrow{\text{res}} & \Lambda[G] & \xrightarrow{\pi} & N & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & \Lambda & \longrightarrow & U^* & \longrightarrow & \bigoplus_{i=1}^{3} \Lambda[G_i] & \longrightarrow & 0
\end{array}
$$

The claim follows from the associated commutative diagram in cohomology:

$$
\begin{array}{ccc}
H^n(\Gamma, N^*) & & \\
\downarrow{\scriptstyle \partial} & & \\
H^{n+1}(\Gamma, N) & \xrightarrow{\partial_r} & H^{n+2}F \\
\downarrow{\scriptstyle f} & & \| \\
\bigoplus_{i=1}^{3} H^{n+1}L_i & \longrightarrow & H^{n+2}F
\end{array}
$$

□

COROLLARY 2. *There are natural isomorphisms*:

$$\mathscr{H}_n(4) \simeq \operatorname{Im}(\partial : H^n(\Gamma, N^*) \longrightarrow H^{n+1}(\Gamma, N)) \simeq \mathscr{H}^n(4).$$

Proof. The cohomology sequences associated to (6) and to (7) intersect at $H^{n+1}(\Gamma, N)$:

$$\begin{array}{ccccccc}
 & & & & \bigoplus_{i=1}^{3} H^n L_i & & \\
 & & & & \big\downarrow{\scriptstyle g} & & \\
 & & & & H^n(\Gamma, N^*) & & \\
 & & & & \big\downarrow{\scriptstyle \partial} & & \\
H^{n+1}F & \xrightarrow{\text{res}} & H^{n+1}M & \longrightarrow & H^{n+1}(\Gamma, N) & \xrightarrow{\partial_r} & H^{n+2}F \\
 & & & & \big\downarrow{\scriptstyle f} & & \\
 & & & & \bigoplus_{i=1}^{3} H^{n+1} L_i & &
\end{array}$$

The lemma of the 700th then shows that the homology of the sequence

$$\bigoplus_{i=1}^{3} H^n L_i \xrightarrow{g} H^n(\Gamma, N^*) \xrightarrow{\partial_r \cdot \partial} H^{n+2}F,$$

which is isomorphic to $\operatorname{Im}(\partial)$ since $\partial_r \cdot \partial = 0$, is canonically isomorphic to the homology of

$$H^{n+1}F \xrightarrow{\text{res}} H^{n+1}M \xrightarrow{f} \bigoplus_{i=1}^{3} H^{n+1}L_i,$$

which is $\mathscr{H}_n(4)$ since f is the direct sum of corestriction maps. The second part of the corollary follows by the same arguments, letting the cohomology sequences associated to (9) and (8) intersect at $H^n(\Gamma, N^*)$.

PROPOSITION 1. *For all integer $n \geq 0$, there are natural isomorphisms*:

$$\begin{aligned}
\mathscr{H}_n(3) \simeq \mathscr{H}^n(3) &\simeq \operatorname{Coker}(\pi + h : H^n M \oplus (G \otimes H^n F) \longrightarrow H^n(\Gamma, N)) \\
&\simeq \operatorname{Ker}(i \oplus h^* : H^{n+1}(\Gamma, N^*) \longrightarrow H^{n+1}M \oplus (X \otimes H^{n+1}F)).
\end{aligned}$$

Proof. We let the cohomology exact sequences associated to (6) and to (10) intersect at $H^n(\Gamma, N)$:

$$\begin{array}{ccccccc}
 & & G \otimes H^n F & & & & \\
 & & \downarrow{\scriptstyle h} & & & & \\
H^n M & \xrightarrow{\pi} & H^n(\Gamma, N) & \xrightarrow{\partial_r} & H^{n+1}F & \xrightarrow{\text{res}} & H^{n+1}M \\
 & & \downarrow{\scriptstyle \text{cor}} & & & & \\
 & & H^n F & & & & \\
 & & \downarrow & & & & \\
 & & G \otimes H^{n+1}F & & & &
\end{array}$$

The lemma of the 700th then yields a natural isomorphism between the homology of

$$H^n M \xrightarrow{\text{cor}\ \pi} H^n F \longrightarrow G \otimes H^{n+1} F,$$

which is $\mathscr{H}^n(3)$, the homology of

$$G \otimes H^n F \xrightarrow{\partial_r\ h} H^{n+1} F \xrightarrow{\text{res}} H^{n+1} M$$

and the cokernel of $\pi + h : H^n M \oplus (G \otimes H^n F) \to H^n(\Gamma, N)$. A straightforward computation shows that $\partial_r \cdot h = \gamma_n$, provided G is identified to X by means of the unique non-degenerate symmetric bilinear form on G, i.e. by letting $\sigma_i = (a_i)$. Arguing similarly, using the cohomology exact sequences associated to (9) and to (11) intersecting at $H^{n+1}(\Gamma, N^*)$, one sees further that $\mathscr{H}_n(3)$ and $\mathscr{H}^n(3)$ are also isomorphic to the kernel of $i \oplus h^*$. □

PROPOSITION 2. *For every integer* $n \geq 0$, *if* $\mathscr{H}_n(3) = \mathscr{H}^n(3) = 0$, *then* $\mathscr{H}_n(1) = \mathscr{H}^n(1) = 0$.

Proof. From the first diagram in the proof of lemma 1, we get a commutative diagram:

$$\begin{array}{ccccc}
 & & H^n F & = & H^n F \\
 & & {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \text{res}} \\
H^n(\Gamma, N) & \xrightarrow{f} & \bigoplus_{i=1}^{3} H^n L_i & \xrightarrow{g} & H^n(\Gamma, N^*) \\
 & & & & \downarrow{\scriptstyle h^*} \\
 & & & & X \otimes H^n F
\end{array}$$

which yields an exact sequence:

$$H^n(\Gamma, N) \oplus H^n F \xrightarrow{f+k} \bigoplus_{i=1}^{3} H^n L_i \xrightarrow{h^* \; g} X \otimes H^n F.$$

As we already noted in the proof of corollary 1, $h^* \cdot g = \beta_n$. Now, the preceding proposition shows that if $\mathscr{H}_n(3) = \mathscr{H}^n(3) = 0$, then the map

$$\pi + h : H^n M \oplus (G \otimes H^n F) \longrightarrow H^n(\Gamma, N)$$

is surjective; therefore, the image of $H^n M \oplus (G \otimes H^n F) \oplus H^n F$ in $\oplus_{i=1}^3 H^n L_i$ under the map $(f+k) \cdot [(\pi + h) \oplus I]$ is the same as the image of $f+k$, which is the kernel of β_n. A direct verification shows that

$$(f+k) \cdot [(\pi + h) \oplus I](u, \sigma_1 \otimes f_1 + \sigma_2 \otimes f_2 + \sigma_3 \otimes f_3, v)$$

$$= \left(N_{M/L_i}(u) + \left(v + \sum_{j \neq i} f_j \right)_{L_i} \right)_{1 \leq i \leq 3}$$

Therefore, the image of this map is the same as the image of α_n, hence $\mathscr{H}_n(1) = 0$.

In order to show that $\mathscr{H}^n(1) = 0$, one argues similarly, starting with the following commutative diagram, also derived from the first diagram in the proof of lemma 1:

$$\begin{array}{ccccc}
G \otimes H^{n+1}F & & & & \\
\downarrow h & & & & \\
H^{n+1}(\Gamma, N) & \xrightarrow{f} & \bigoplus_{i=1}^{3} H^{n+1} L_i & \xrightarrow{g} & H^{n+1}(\Gamma, N^*) \\
\downarrow \mathrm{cor} & & \downarrow l & & \\
H^{n+1}F & = & H^{n+1}F & &
\end{array}$$

and using the hypothesis that $\mathscr{H}_n(3) = \mathscr{H}^n(3) = 0$ to see that the map

$$(i \oplus h^*) \oplus I : H^{n+1}(\Gamma, N^*) \oplus H^{n+1}F \longrightarrow H^{n+1}M \oplus (X \otimes H^{n+1}F) \oplus H^{n+1}F$$

is injective, by the preceding proposition. □

PROPOSITION 3. *For every integer* $n \geq 0$, *if* $\mathscr{H}_n(3) = \mathscr{H}^n(3) = 0$, *then* $\mathscr{H}_n(5) = \mathscr{H}^n(5) = 0$.

Proof. The diagram involving U^* in the proof of lemma 2 yields in cohomology:

$$\begin{array}{ccccc}
H^{n+1}M & \longrightarrow & H^{n+1}(\Gamma, N) & \xrightarrow{\partial_r} & H^{n+2}F \\
 & & \Big\downarrow f & & \Big\| \\
 & & \bigoplus_{i=1}^{3} H^{n+1}L_i & \xrightarrow{\partial_i} & H^{n+2}F \\
 & & \Big\downarrow g & & \\
 & & H^{n+1}(\Gamma, N^*) & &
\end{array}$$

hence there is an exact sequence:

$$H^{n+1}M \xrightarrow{\delta_n} \bigoplus_{i=1}^{3} H^{n+1}L_i \xrightarrow{g \oplus \partial_i'} H^{n+1}(\Gamma, N^*) \oplus H^{n+2}F.$$

Proposition 1 shows that $i \oplus h^* : H^{n+1}(\Gamma, N^*) \to H^{n+1}M \oplus (X \otimes H^{n+1}F)$ is injective. It then suffices to see that $[(i \oplus h^*) \oplus I] \cdot (g \oplus \partial_r') = \varepsilon_n$ to complete the proof that $\mathscr{H}_n(5) = 0$. To prove that $\mathscr{H}^n(5) = 0$, one uses the same arguments, starting with the first diagram in the proof of lemma 2 and using proposition 1 to see that the map $\pi + h : H^nM \oplus (G \otimes H^nF) \to H^n(\Gamma, N)$ is surjective. □

Theorem A follows from corollaries 1 and 2, and from propositions 1, 2 and 3.

In closing this section, we observe that Theorem A can also be proved directly by elementary (but tedious) computations using the exactness of sequence (1) for quadratic extensions instead of cohomology computations. For example—and because it yields additional information—we note that the isomorphism $\mathscr{H}_n(2) \simeq \mathscr{H}^n(2)$ can also be proved by using the following commutative diagram, in which we denote simply $A(k) = H^kM \oplus (H^kF)^3$, for $k = n, n+1$:

$$\begin{array}{ccccccccc}
A(n) & \xrightarrow{\alpha_n} & \bigoplus_{i=1}^{3} H^nL_i & \xrightarrow{\beta_n} & X \otimes H^nF & \xrightarrow{\gamma_n} & H^{n+1}F & \xrightarrow{\text{res}} & H^{n+1}M \\
\Big\downarrow I+0 & & \Big\downarrow \Sigma\,\text{cor} = f & & \Big\downarrow \Delta_n & & \Big\downarrow \oplus\,\text{res} = f^* & & \Big\downarrow I \oplus 0 \\
H^nM & \xrightarrow{\text{cor}} & H^nF & \xrightarrow{\gamma^n} & G \otimes H^{n+1}F & \xrightarrow{\beta^n} & \bigoplus_{i=1}^{3} H^{n+1}L_i & \xrightarrow{\alpha^n} & A(n+1)
\end{array}$$

where the map Δ_n is defined by

$$\Delta_n\left(\sum_{i=1}^{3} (a_i) \otimes f_i\right) = \sum_{i=1}^{3} \sigma_i \otimes (a_i) \cdot f_i + \sum_{i=1}^{3} \sigma_i \otimes (a_i) \cdot \left(\sum_{j=1}^{3} f_j\right).$$

(This map Δ_n is induced by the bilinear map $X \times H^nF \to G \otimes H^{n+1}F$ which carries (χ, f) to $\Sigma_{i=1}^3 (\langle \chi, \sigma_i \rangle \sigma_i) \otimes ((a_i) \cdot f)$). From this diagram, isomorphisms $\mathscr{H}_n(1) \simeq \mathscr{H}^n(3)$ and $\mathscr{H}_n(3) \simeq \mathscr{H}^n(1)$ can also be derived; therefore, we have in fact

$$\mathscr{H}_n(1) \simeq \mathscr{H}^n(3) \simeq \mathscr{H}_n(3) \simeq \mathscr{H}^n(1).$$

2. Proof of Theorem B

In view of Theorem A, vanishing of one of the groups $\mathscr{H}_n(2)$, $\mathscr{H}^n(2)$, $\mathscr{H}_n(4)$, $\mathscr{H}^n(4)$ and one of the groups $\mathscr{H}_n(3)$, $\mathscr{H}^n(3)$ for some $n \geq 0$ implies that the sequences S_n and S^n are both exact. Of course, the difficulty of proving such a result increases very rapidly with n. While our main results concern the case where $n = 2$ or 3, we first review the case $n = 0$ (which is trivial) and $n = 1$.

2.1. $n = 0$

Since $\beta_0 = 0$ and $\gamma_0 : X \otimes H^0F \to H^1F$ is the inclusion of $X = X \otimes H^0F$ in H^1F, exactness of the sequence:

$$\bigoplus_{i=1}^{3} H^0L_i \xrightarrow{\beta_0} X \otimes H^0F \xrightarrow{\gamma_0} H^1F \xrightarrow{\text{res}} H^1M$$

readily follows from the definition of X. This shows $\mathscr{H}_0(2) = \mathscr{H}_0(3) = 0$, hence S_0 and S^0 are both exact.

2.2. $n = 1$

We first show $\mathscr{H}_1(2) = 0$: let $\Sigma_{i=1}^3 (a_i) \otimes (x_i) \in \operatorname{Ker}(\gamma_1)$; we then have the following relation between quaternion algebras:

$$(a_1, x_1)_F \otimes (a_2, x_2)_F \otimes (a_3, x_3)_F = 0 \quad \text{in } H^2F = {}_2\text{Br}(F),$$

or, taking into account the fact that $(a_1) + (a_2) + (a_3) = 0$:

$$(a_1, x_1x_3)_F = (a_2, x_2x_3)_F.$$

The "common slot lemma" [22, p. 267] or [1, Lemma 1.7] yields an element $y \in F^\times$ such that

$$(a_1, x_1x_3)_F = (a_1, y)_F = (a_2, y)_F = (a_2, x_2x_3)_F.$$

From these relations, it follows that

$$(a_1, x_1x_3y)_F = (a_3, y)_F = (a_2, x_2x_3y)_F = 0,$$

hence $x_1x_3y = N_{L_1/F}(l_1)$, $x_2x_3y = N_{L_2/F}(l_2)$ and $y = N_{L_3/F}(l_3)$ for some $l_i \in L_i^\times$ $(i = 1, 2, 3)$, and

$$\sum_{i=1}^{3} (a_i) \otimes (x_i) = \sum_{i=1}^{3} (a_i) \otimes N_{L_i/F}(l_i) = \beta_1((l_1), (l_2), (l_3)).$$

This proves $\mathscr{H}_1(2) = 0$.

To complete the proof that S_1 and S^1 are both exact, it now suffices to show $\mathscr{H}_1(3) = 0$. Let $u \in H^2F = {}_2\mathrm{Br}(F)$ be such that $u_M = 0$. Then u_{L_1} is split by $M = L_1(\sqrt{a_2})$, hence it is a quaternion algebra:

$$u_{L_1} = (a_2, v)_{L_1}$$

for some $v \in L_1^\times$. From $N_{L_1/F}(u_{L_1}) = 0$ it follows:

$$(a_2, N_{L_1/F}(v))_F = 0.$$

Therefore, by [5, 2.13] or [23, Cor. 2.10] (or lemma 3 below), there exists $r \in F^\times$ such that

$$(a_2, v)_{L_1} = (a_2, r)_F \otimes L_1.$$

Then $u - (a_2, r)_F$ is split by L_1, hence there exists $t \in F^\times$ such that $u - (a_2, r)_F = (a_1, t)_F$, whence

$$u = (a_1, t)_F \otimes (a_2, r)_F = \gamma_1((a_1) \otimes (t) + (a_2) \otimes (r)).$$

This shows $\mathscr{H}_1(3) = 0$ and completes the proof that S_1 and S^1 are both exact.

2.3. $n \geq 2$

In order to prove exactness of the sequences S_2 and S^2, we will need to switch from Galois cohomology to Milnor's K-theory and Witt rings of quadratic forms. For any field F, we simply denote by K_nF the Milnor K-groups of the field F, as defined in [15], and by WF the Witt ring of F (if char. $F \neq 2$). The n-th power of the fundamental ideal IF of WF is denoted by I^nF, and we let $\bar{I}^nF = I^nF/I^{n+1}F$ and $k_nF = K_nF/2K_nF$.

The element in K_1F corresponding to $a \in F^\times$ is denoted by $\{a\}$, and for $a_1, \ldots, a_n \in F^\times$ we let

$$\{a_1, \ldots, a_n\} = \{a_1\} \ldots \{a_n\} \in K_nF.$$

Similarly, for $a \in F^\times$ we denote by $\langle\langle a \rangle\rangle$ the (isometry class of the) quadratic form $\langle 1, -a \rangle = X^2 - aY^2$, and also its image in the Witt ring WF. If $a_1, \ldots, a_n \in F^\times$ we let

$$\langle\langle a_1, \ldots, a_n \rangle\rangle = \langle\langle a_1 \rangle\rangle \ldots \langle\langle a_n \rangle\rangle \in I^nF$$

and

$$(a_1, \ldots, a_n) = (a_1) \ldots (a_n) \in H^nF.$$

We recall the homomorphisms:

$$\begin{array}{ccc} k_nF & \xrightarrow{R_n} & H^nF \\ & {\scriptstyle s_n}\searrow \quad \nearrow{\scriptstyle e_n} & \\ & \bar{I}^nF & \end{array}$$

defined by: $R_n(\{a_1, \ldots, a_n\}) = (a_1, \ldots, a_n)$ and $s_n(\{a_1, \ldots, a_n\}) = \langle\langle a_1, \ldots, a_n \rangle\rangle + I^{n+1}F$. The existence of the map e_n for which the triangle commutes is proven for all $n \leq 4$. If $n = 2$, it is given by the Clifford invariant: see [10, Chap. 5]; for $n = 3$ it was proved by Arason [1], for $n = 4$ by Jacob–Rost [8] and Szyjewski [21].

The map R_n is known as the *residue norm homomorphism*; it is an isomorphism for $n \leq 3$. This result was proved for $n = 2$ by Merkurjev [11], for $n = 3$ by Merkurjev–Suslin [14] and by Rost [18].

The map s_n is surjective for all n; since R_n is an isomorphism and e_n is defined for $n \leq 3$, it follows that s_n is an isomorphism for $n \leq 3$.

Even though e_n is not known to be well-defined for $n \geq 5$, it is well-defined on Pfister forms: for $a_1, \ldots, a_n$; and $b_1, \ldots, b_n \in F^\times$, we have $\langle\langle a_1, \ldots, a_n \rangle\rangle =$

$\langle\langle b_1, \ldots, b_n \rangle\rangle$ in WF if and only if $\{a_1, \ldots, a_n\} = \{b_1, \ldots, b_n\}$ in k_nF, and any of these relations implies: $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ in H^nF: see [1, Satz 1.6] and [4, Theorem 3.2].

If $K = F(\sqrt{a})$ is a quadratic extension of a field F of characteristic not 2, there are zero-sequences analogous to (1):

$$\cdots \longrightarrow k_{n-1}F \xrightarrow{\{a\}} k_nF \longrightarrow k_nK \xrightarrow{N} k_nF \xrightarrow{\{a\}} k_{n+1}F \longrightarrow \cdots \tag{12}$$

and:

$$\cdots \longrightarrow \bar{I}^{n-1}F \xrightarrow{\langle\langle a\rangle\rangle} \bar{I}^nF \longrightarrow \bar{I}^nK \xrightarrow{N} \bar{I}^nF \xrightarrow{\langle\langle a\rangle\rangle} \bar{I}^{n+1}F \longrightarrow \cdots \tag{13}$$

Since these sequences are related to (1) by the homomorphisms R_n and e_n (for $n \leq 4$), bijectivity of R_n and e_n for $n \leq 3$ implies that sequences (12) and (13) are exact when they are truncated at k_4F and $\bar{I}^4F$ respectively; thus:

$$0 \longrightarrow k_0F \longrightarrow k_0K \longrightarrow k_0F \longrightarrow \cdots \longrightarrow k_3F \longrightarrow k_3K \longrightarrow k_3F \longrightarrow k_4F \tag{14}$$

and

$$0 \longrightarrow \bar{I}^0F \longrightarrow \bar{I}^0K \longrightarrow \bar{I}^0F \longrightarrow \cdots \longrightarrow \bar{I}^3F \longrightarrow \bar{I}^3K \longrightarrow \bar{I}^3F \longrightarrow \bar{I}^4F \tag{15}$$

are exact.

The following lemma generalizes one of the steps in the proof of $\mathscr{H}_1(3) = 0$ above:

LEMMA 3. *Let K/F be a quadratic extension of fields of characteristic not* 2, *let $k \in K^\times$ and let $f_1, \ldots, f_{n-1} \in F^\times$. If $\{f_1, \ldots, f_{n-1}, N_{K/F}(k)\} = 0$ in k_nF, then there exists $f_n \in F^\times$ such that $\{f_1, \ldots, f_{n-1}, k\} = \{f_1, \ldots, f_n\}_K$ in k_nK.*

Proof. If $n = 1$, the lemma is another way of stating that sequence (14) is exact at k_1K.

For $n = 2$, the lemma readily follows from [5, 2.13] or [23, Cor. 2.10] or [14, Lemma 2.9]. We include a proof for the reader's convenience: if f_1 in a square in K, then $\{f_1, k\} = 0$ in k_2K and we may choose $f_2 = 1$ in this case. We may thus assume $K(\sqrt{f_1})$ is a field. It is then an elementary abelian extension of F. Let σ (resp. τ) be the non-trivial automorphism which leaves $F(\sqrt{f_1})$ (resp. K) elementwise invariant. The hypothesis that $\{f_1, N_{K/F}(k)\} = 0$ in k_2F means that $k\sigma(k) = x\tau(x)$ for some $x \in F(\sqrt{f_1})$. Certainly, $x + \tau(x)$ and $k + \sigma(k)$ are not both zero since the preceding equation would then yield: $k^2 = x^2$, which is impossible since $k \in K$ and $x \in F(\sqrt{f_1})$. Changing x into $-x$ if necessary, we may thus assume $k + \sigma(k) + x + \tau(x) \neq 0$. We may then choose $f_2 = k + \sigma(k) + x + \tau(x)$, since the relation:

$$(1 + xk^{-1}) \cdot \tau(1 + xk^{-1}) = f_2 k^{-1}$$

shows that $f_2 k^{-1}$ is a norm from $K(\sqrt{f_1})$ to K, hence $\{f_1, f_2 k^{-1}\} = 0$ in $k_2 K$.

In the case where $n \geq 3$, the idea is to reduce to the case $n = 2$, as follows: switching to quadratic forms, the hypothesis yields: $\langle\langle f_1, \ldots, f_{n-1}, N_{K/F}(k)\rangle\rangle = 0$ in WF; therefore,

$$\langle\langle f_1, \ldots, f_{n-1}\rangle\rangle = \langle N_{K/F}(k)\rangle \cdot \langle\langle f_1, \ldots, f_{n-1}\rangle\rangle,$$

hence, by [10, Cor. 10.1.7], $N_{K/F}(k)$ is represented by $\langle\langle f_1, \ldots, f_{n-1}\rangle\rangle$. If $N_{K/F}(k) \in F^{\times 2}$, then there exists $f_n \in F^\times$ such that $k \equiv f_n \bmod K^{\times 2}$, by the exactness of (1) at $H^1 K$, hence $\{f_1, \ldots, f_{n-1}, k\} = \{f_1, \ldots, f_n\}_K$ in $k_n K$. If $N_{N/F}(k) \notin F^{\times 2}$, let

$$N_{N/F}(k) = x^2 - f' \tag{16}$$

where $x, f' \in F$ and $-f'$ is represented by the pure subform of $\langle\langle f_1, \ldots, f_{n-1}\rangle\rangle$. By [10, Prop. 10.1.5], we then have

$$\langle\langle f_1, \ldots, f_{n-1}\rangle\rangle = \langle\langle f'_1, \ldots, f'_{n-2}, f'\rangle\rangle \tag{17}$$

for some $f'_1, \ldots, f'_{n-2} \in F^\times$. Now, equation (16) yields: $\{f', N_{K/F}(k)\} = 0$ in $k_2 F$, hence, by the preceding part of the proof (when $n = 2$), there exists $f_n \in F^\times$ such that

$$\{f', k\} = \{f', f_n\}_K.$$

Multiplying both sides of this equation by $\{f'_1, \ldots, f'_{n-2}\}$, we get:

$$\{f'_1, \ldots, f'_{n-2}, f'\}_K \cdot \{k\} = \{f'_1, \ldots, f'_{n-2}, f'\}_K \cdot \{f_n\}_K \quad \text{in } k_n K,$$

hence, by (17):

$$\{f_1, \ldots, f_{n-1}, k\} = \{f_1, \ldots, f_n\}_K. \qquad \square$$

We will also need the following characterization of the elements in K_nK which are fixed under the Galois group of K/F:

LEMMA 4. *Let* $\bar{\ }$ *denote the non-trivial automorphism of the quadratic extension* $K = F(\sqrt{a})$. *For* $n \le 3$, *every element* $u \in K_nK$ *such that* $\bar{u} = u$ *can be written in the form*:

$$u = v_K + \{\sqrt{a}\} \cdot w_K$$

for some $v \in K_nF$, $w \in K_{n-1}F$.

Proof. From $\bar{u} = u$, it follows that $N_{K/F}(u)_K = u + \bar{u} = 2u$, hence the image of $N_{K/F}(u)$ in k_nF is the kernel of the extension of scalars map. By the exactness of sequence (14), it follows that

$$N_{K/F}(u) = \{a\} \cdot t + 2s$$

for some $t \in K_{n-1}F$, $s \in K_nF$, hence, extending scalars to K:

$$2u = 2\{\sqrt{a}\} \cdot t_K + 2s_K.$$

Since the 2-torsion subgroup in K_nK is $\{-1\} \cdot K_{n-1}K$, by [13, Theorem 14.2] and [14, Prop. 6.1], it follows that

$$u = \{\sqrt{a}\} \cdot t_K + s_K + \{-1\} \cdot x \tag{18}$$

for some $x \in K_{n-1}K$. Since $\{-1\} = \{\sqrt{a}\} - \{-\sqrt{a}\}$, we have:

$$\begin{aligned}\{-1\} \cdot x &= \{\sqrt{a}\} \cdot (x + \bar{x}) - (\{-\sqrt{a}\} \cdot x + \{\sqrt{a}\} \cdot \bar{x})\\ &= \{\sqrt{a}\} \cdot N_{K/F}(x)_K - N_{K/F}(\{-\sqrt{a}\} \cdot x)_K.\end{aligned}$$

Substituting for $\{-1\} \cdot x$ in equation (18), we get:

$$u = \{\sqrt{a}\} \cdot (t + N_{K/F}(x))_K + (s - N_{K/F}(\{-\sqrt{a}\} \cdot x))_K. \qquad \square$$

We now return to the notation set up in the introduction, and aim to prove $\mathscr{H}_n(3) = 0$ for $n = 2, 3$. The main idea of the proof of $\mathscr{H}_1(3) = 0$ above was to extend scalars from F to L_1 and work back from there. More generally, this idea yields the following reduction:

LEMMA 5. *Let n be an arbitrary positive integer. If every* $v \in H^n L_1$ *such that*

$$(a_2) \cdot N_{L_1/F}(v) = 0$$

can be written in the form

$$v = r_{L_1} + N_{M/L_1}(s)$$

for some $r \in H^n F$, $s \in H^n M$, *then* $\mathscr{H}_n(3) = 0$.

(The converse is also true, but will not be needed).

Proof. Let $u \in H^{n+1}F$ be such that $u_M = 0$; then u_{L_1} is split by $M = L_1(\sqrt{a_2})$, hence by the exactness of sequence (1) for the quadratic extension M/L_1, we have:

$$u_{L_1} = (a_2)_{L_1} \cdot v$$

for some $v \in H^n L_1$. From $N_{L_1/F}(u_{L_1}) = 0$, it follows that $(a_2) \cdot N_{L_1/F}(v) = 0$. The hypothesis then yields $r \in H^n F$, $s \in H^n M$ such that

$$v = r_{L_1} + N_{M/L_1}(s).$$

Since $(a_2) \cdot N_{M/L_1}(s) = 0$, we get:

$$u_{L_1} = (a_2)_{L_1} \cdot r_{L_1},$$

hence $u - (a_2) \cdot r$ is split by L_1. Using sequence (1) for L_1/F, we have $u - (a_2) \cdot r = (a_1) \cdot t$ for some $t \in H^n F$, hence

$$u = (a_1) \cdot t + (a_2) \cdot r = \gamma_n((a_1) \otimes t + (a_2) \otimes r). \qquad \square$$

PROPOSITION 4. $\mathscr{H}_n(3) = 0$ *for* $n = 2, 3$.

Proof. We first show that it suffices to prove, for $n = 2, 3$:

(*) Every $v \in K_n L_1$ such that $N_{L_1/F}(v) \equiv N_{L_2/F}(w) \bmod 2K_n F$ for some $w \in K_n L_2$ can be written in the form: $v = r_{L_1} + N_{M/L_1}(s)$ for some $r \in K_n F$, $s \in K_n M$.

Indeed, if $v' \in H^n L_1$ is such that $(a_2) \cdot N_{L_1/F}(v') = 0$, then by surjectivity of R_n for $n = 2, 3$ one can find $v \in K_n L_1$ such that $R_n(v) = v'$. Moreover, bijectivity of R_n for

$n = 2, 3$ and commutativity of the diagram

$$\begin{array}{ccccc} k_nL_2 & \xrightarrow{N} & k_nF & & \\ {\scriptstyle R_n}\downarrow & & \downarrow{\scriptstyle R_n} & & \\ H^nL_2 & \xrightarrow{N} & H^nF & \xrightarrow{(a_2)\cdot} & H^{n+1}F \end{array}$$

whose bottom row is exact, shows that $N_{L_1/F}(v) \equiv N_{L_2/F}(w) \bmod 2K_nF$ for some $w \in K_nL_2$. By $(*)$, it follows that

$$v' = R_n(r)_{L_1} + N_{M/L_1}(R_n(s))$$

for some $r \in K_nF$, $s \in K_nM$, hence lemma 5 shows that $\mathscr{H}_n(3) = 0$. Thus, it suffices to prove $(*)$.

In order to do that, we first reduce to the case where $\bar{v} = v$. Since $N_{L_2/F}(u) = 2u$ for every $u \in K_nF$, we may assume, after adding to w an element in K_nF, that

$$N_{L_1/F}(v) = N_{L_2/F}(w). \tag{19}$$

Then,

$$N_{M/L_3}(v_M - w_M) = N_{L_1/F}(v)_{L_3} - N_{L_2/F}(w)_{L_3} = 0.$$

From Hilbert's theorem 90 for K_n ([7, Satz 90] for $n = 1$, [13, Theorem 14.1] for $n = 2$ and [14, Theorem 4.1] for $n = 3$), it follows that

$$u_M - w_M = t - \sigma_3(t)$$

for some $t \in K_nM$. We then have

$$N_{M/L_1}(\sigma_3(t)) = N_{M/L_1}(t) - 2v + N_{L_2/F}(w)_{L_1}$$

hence, by (19):

$$N_{M/L_1}(\sigma_3(t)) = N_{M/L_1}(t) + \bar{v} - v.$$

Substituting $v - N_{M/L_1}(t)$ for v, we may thus assume $\bar{v} = v$.

Applying lemma 4, we now get:

$$v = x_{L_1} + \{\sqrt{a_1}\} \cdot y_{L_1}$$

for some $x \in K_nF$, $y \in K_{n-1}F$. From (19), it follows that $\{a_2\} \cdot N_{L_1/F}(v) \in 2K_{n+1}F$, hence

$$\{a_2, -a_1\} \cdot y \in 2K_{n+1}F. \tag{20}$$

If $\{a_2, -a_1\} \in 2K_2F$ (i.e. if the quaternion algebra $D = (a_2, -a_1)_F$ is split), then it follows from lemma 3 that $\{a_2, \sqrt{a_1}\} \equiv \{a_2, f\}$ mod $2K_2L_1$ for some $f \in F^\times$. Therefore, $\{a_2\}_{L_1} \cdot (\{\sqrt{a_1}\} - \{f\}_{L_1}) \in 2K_2L_1$ and, by the exactness of sequence (14), $\{\sqrt{a_1}\} - \{f\}_{L_1} = N_{M/L_1}(z)$ for some $z \in K_1M$. We then have $\{\sqrt{a_1}\} \cdot y_{L_1} = (\{f\} \cdot y)_{L_1} + N_{M/L_1}(y_M z)$, hence

$$v = (x + \{f\} \cdot z)_{L_1} + N_{M/L_1}(y_M z),$$

and the proof is complete in this case.

Suppose then that the quaternion algebra $D = (a_2, -a_1)_F$ is not split. If $n = 2$, then $y = \{g\}$ for some $g \in F^\times$ and from relation (20) it follows by lemma 3 that $\{a_2, \sqrt{a_1}, g\} \equiv \{a_2, b, g\}$ mod $2K_3L_1$ for some $b \in F^\times$, hence, by the exactness of sequence (14),

$$\{\sqrt{a_1}, g\} = \{b, g\}_{L_1} + N_{M/L_1}(z)$$

for some $z \in K_3M$. Since the left-hand side is $\{\sqrt{a_1}\} \cdot y_{L_1}$, we get:

$$v = (x + \{b, g\})_{L_1} + N_{M/L_1}(z),$$

and the proof is complete for $n = 2$.

Finally, we consider the case where $n = 3$ and D is a division algebra. By [12, Theorem 2], it then follows from (20) that y lies in the image of the reduced norm map Nrd: $K_2D \to K_2F$. Since K_2D is generated by symbols of the type $\{f, d\}$ where $f \in F^\times$ and $d \in D^\times$ (see [16], [17]) and since the reduced norm of such a symbol is $\{f, \mathrm{Nrd}(d)\}$, the element y has the form: $y = \Sigma_i \{f_i, \mathrm{Nrd}(d_i)\}$. For all i, we have $\{a_2, -a_1, \mathrm{Nrd}(d_i)\} \in 2K_3F$, hence

$$\{a_2, -a_1, f_i, \mathrm{Nrd}(d_i)\} \in 2K_4F.$$

Lemma 3 then yields an element $g_i \in F^\times$ such that

$$\{a_2, \sqrt{a_1}, f_i, \mathrm{Nrd}(d_i)\} \equiv \{a_2, g_i, f_i, \mathrm{Nrd}(d_i)\}_{L_1} \mod 2K_4L_1.$$

By the exactness of sequence (14), we get:

$$\{\sqrt{a_1}, f_i, \mathrm{Nrd}\,(d_i)\} = \{g_i, f_i, \mathrm{Nrd}\,(d_i)\} + N_{M/L_1}(z_i)$$

for some $z_i \in K_3M$. Summing over i, we obtain:

$$\{\sqrt{a_1}\} \cdot y_{L_1} = \sum_i (\{g_i, f_i, \mathrm{Nrd}\,(d_i)\}_{L_1} + N_{M/L_1}(z_i)),$$

hence

$$v = \left(x + \sum_i \{g_i, f_i, \mathrm{Nrd}\,(d_i)\}\right)_{L_1} + N_{M/L_1}\left(\sum_i z_i\right). \qquad \square$$

To complete the proof of Theorem B, it now suffices to show that $\mathscr{H}_2(2) = 0$. We first derive from our proof of $\mathscr{H}_1(2) = 0$ above the following general statement, which shows that $\mathscr{H}_n(2) = 0$ follows from a kind of "common slot lemma":

LEMMA 6. *Let $n \geq 1$. If for all $s, t \in H^nF$ such that $(a_1) \cdot s = (a_2) \cdot t$ there exists $u \in H^nF$ such that*

$$(a_1) \cdot s = (a_1) \cdot u = (a_2) \cdot u = (a_2) \cdot t,$$

then $\mathscr{H}_n(2) = 0$.

(The converse is also true, but will not be needed).

Proof. Let $\Sigma_{i=1}^3 (a_i) \otimes f_i \in X \otimes H^nF$ be in the kernel of γ_n; then

$$(a_1) \cdot (f_1 + f_3) = (a_2) \cdot (f_2 + f_3),$$

hence, by hypothesis, there exists $u \in H^nF$ such that

$$(a_1) \cdot (f_1 + f_3) = (a_1) \cdot u = (a_2) \cdot u = (a_2) \cdot (f_2 + f_3).$$

From these relations, it follows that

$$(a_1) \cdot (f_1 + f_3 + u) = (a_3) \cdot u = (a_2) \cdot (f_2 + f_3 + u) = 0,$$

hence $f_1+f_3+u=N_{L_1/F}(l_1)$, $f_2+f_3+u=N_{L_2/F}(l_2)$ and $u=N_{L_3/F}(l_3)$ for some $l_i \in H^nL_i$ $(i=1, 2, 3)$, and

$$\sum_{i=1}^{3}(a_i)\otimes f_i=\sum_{i=1}^{3}(a_i)\otimes N_{L_i/F}(l_i)=\beta_n(l_1, l_2, l_3).$$

This proves $\mathscr{H}_n(2)=0$. □

We now show that the hypothesis of the preceding lemma holds for $n=2$, at least when s is a single symbol:

LEMMA 7. *If $s=(b, c)\in H^2F$ and $t\in H^2F$ are such that $(a_1)\cdot s=(a_2)\cdot t$, then there exists $u\in H^2F$ such that*

$$(a_1)\cdot s=(a_1)\cdot u=(a_2)\cdot u=(a_2)\cdot t.$$

Proof. Under the isomorphism e_3, the hypothesis translates to: $\langle\langle a_1, b, c\rangle\rangle_{L_2}=0$ in $\bar{I}^3L_2$, hence by the "Hauptsatz" of Arason and Pfister [10, Theorem 10.3.1], $\langle\langle a_1, b, c\rangle\rangle_{L_2}$ is hyperbolic. By [10, Theorem 7.3.2], it follows that

$$\langle\langle a_1, b, c\rangle\rangle=\langle\langle a_2, b', c'\rangle\rangle$$

for some $b', c'\in F^\times$. Arason's "common slot lemma" [1, Lemma 1.7] yields an element $u_1\in F^\times$ such that

$$\langle\langle a_1, b, c\rangle\rangle=\langle\langle a_1, u_1, c\rangle\rangle \quad\text{and}\quad \langle\langle a_2, b', c'\rangle\rangle=\langle\langle a_2, u_1, c'\rangle\rangle,$$

hence

$$\langle\langle a_1, u_1, c\rangle\rangle=\langle\langle a_2, u_1, c'\rangle\rangle.$$

Applying again the "common slot lemma", we get an element $u_2\in F^\times$ such that

$$\langle\langle a_1, u_1, c\rangle\rangle=\langle\langle a_1, u_1, u_2\rangle\rangle \quad\text{and}\quad \langle\langle a_2, u_1, c'\rangle\rangle=\langle\langle a_2, u_1, u_2\rangle\rangle.$$

Thus,

$$\langle\langle a_1, b, c\rangle\rangle=\langle\langle a_1, u_1, u_2\rangle\rangle=\langle\langle a_2, u_1, u_2\rangle\rangle=\langle\langle a_2, b', c'\rangle\rangle.$$

The element $u=(u_1, u_2)\in H^2F$ therefore satisfies the required conditions. □

More generally, repeated use of Arason's "common slot lemma" [1, Lemma 1.7] shows that the hypothesis of lemma 6 holds in $\bar{I}^n F$ instead of $H^n F$, at least when s is represented by a Pfister form.

We aim to show that the hypothesis of lemma 6 holds in general for $n = 2$, by arguing by induction on the number of terms in a representation of s as a sum of symbols. The initial step is of course lemma 7. In order to carry out the induction step, we shall use the following approach: let $b, c \in F^\times$ and let ϕ denote the 8-dimensional quadratic form:

$$\phi = \langle a_2 \rangle \perp \langle\langle a_1, b, c \rangle\rangle',$$

where $\langle\langle a_1, b, c \rangle\rangle'$ is the pure subform of the Pfister form $\langle\langle a_1, b, c \rangle\rangle$. Explicitly,

$$\phi = a_2 x_0^2 - a_1 x_1^2 - b x_2^2 - c x_3^2 + a_1 b x_4^2 + a_1 c x_5^2 + b c x_6^2 - a_1 b c x_7^2.$$

Let $X = X(\phi)$ be the associated projective quadric and let $F(X)$ denote its function field. Since the pure subform of $\langle\langle a_1, b, c \rangle\rangle$ represents $-a_2$ over $F(X)$, it follows from [10, Prop. 10.1.5] that

$$\langle\langle a_1, b, c \rangle\rangle = \langle\langle a_2, b', c' \rangle\rangle \tag{21}$$

for some $b', c' \in F(X)^\times$. The same arguments show that ϕ is isotropic if and only if there exists $b', c' \in F^\times$ for which (21) holds.

For any field E containing F we denote by $h(E)$ the homology of the sequence:

$$H^2 E \xrightarrow{\gamma^2} G \otimes H^3 E \xrightarrow{\beta^2} \bigoplus_{i=1}^{3} H^3(L_i \otimes E);$$

thus, $h(F) = \mathscr{H}^2(2)$. The main step of the proof is to show:

PROPOSITION 5. *The map $h(F) \to h(F(X))$ induced by the inclusion of F in $F(X)$ is injective.*

Using this proposition, the proof of Theorem B can be completed as follows: in view of lemma 6, we only have to prove that if $s, t \in H^2 F$ are such that $(a_1) \cdot s = (a_2) \cdot t$, then there exists $u \in H^2 F$ such that $(a_1) \cdot s = (a_1) \cdot u = (a_2) \cdot u = (a_2) \cdot t$. Lemma 7 shows that this condition holds if s is a single symbol. Suppose then $s = (b, c) + s'$, where s' is a sum of fewer symbols than s. Extending scalars to $F(X)$, we get from (21):

$$(a_1) \cdot (b, c) = (a_2) \cdot (b', c'),$$

hence $(a_1) \cdot s' = (a_2) \cdot (t + (b', c'))$. By induction, we get $u' \in H^2(F(X))$ such that

$$(a_1) \cdot s' = (a_1) \cdot u' = (a_2) \cdot u'.$$

On the other hand, by lemma 7 we get $u'' \in H^2(F(X))$ such that

$$(a_1) \cdot (b, c) = (a_1) \cdot u'' = (a_2) \cdot u''.$$

Therefore,

$$(a_1) \cdot s = (a_1) \cdot (u' + u'') = (a_2) \cdot (u' + u'') = (a_2) \cdot t. \tag{22}$$

Consider then $\sigma_1 \otimes (a_2) \cdot t + \sigma_2 \otimes (a_1) \cdot s \in G \otimes H^3 F$. Since $(a_1) \cdot s = (a_2) \cdot t$, this element is in the kernel of β^2. On the other hand, equation (22) shows that

$$\sigma_1 \otimes (a_2) \cdot t + \sigma_2 \otimes (a_1) \cdot s = \gamma^2(u' + u'') \in G \otimes H^3(F(X)),$$

hence this element represents the trivial element of $h(F(X))$. By proposition 5, it follows that this element is trivial in $h(F)$, hence there exists $u \in H^2 F$ such that

$$\sigma_1 \otimes (a_2) \cdot t + \sigma_2 \otimes (a_1) \cdot s = \gamma^2(u).$$

From this last relation, it readily follows that $(a_1) \cdot s = (a_1) \cdot u = (a_2) \cdot u = (a_2) \cdot t$, and the proof of Theorem B is complete.

Alternatively, one can repeat the function field construction above to obtain a field Ω containing F such that the natural map $h(F) \to h(\Omega)$ is injective and for every $b, c \in \Omega^\times$ there exist $b', c' \in \Omega^\times$ such that $(a_1) \cdot (b, c) = (a_2) \cdot (b', c')$. Using lemma 7, it readily follows that $h(\Omega) = 0$, hence $h(F) = 0$.

We now proceed to prove proposition 5. Consider the following diagram:

$$\begin{array}{ccccc}
H^2 F & \xrightarrow{\gamma^2} & G \otimes H^3 F & \xrightarrow{\beta^2} & \bigoplus_{i=1}^{3} H^3 L_i \\
\downarrow{\scriptstyle \nu_1} & & \downarrow{\scriptstyle \nu_2} & & \downarrow{\scriptstyle \nu_3} \\
H^2(F(X)) & \xrightarrow{\gamma_X^2} & G \otimes H^3(F(X)) & \xrightarrow{\beta_X^2} & \bigoplus_{i=1}^{3} H^3(L_i(X))
\end{array}$$

where ν_1, ν_2, ν_3 are the natural maps, and define a group A as follows:

$$A = \frac{\{u \in G \otimes H^3 F \mid \nu_2(u) \in \operatorname{Im} \gamma_X^2\}}{\operatorname{Im} \gamma^2} = \operatorname{Ker}(\operatorname{Coker} \gamma^2 \longrightarrow \operatorname{Coker} \gamma_X^2).$$

LEMMA 8. *There is a natural exact sequence*:

$$0 \longrightarrow \operatorname{Ker}(h(F) \longrightarrow h(F(X)) \longrightarrow A \longrightarrow \operatorname{Ker} v_3 \longrightarrow 0.$$

Moreover, $\operatorname{Ker} v_3 = 0$ *if* ϕ *is isotropic and* $\operatorname{Ker} v_3$ *is a group of order* 2 *if* ϕ *is anisotropic.*

Proof. A theorem of Arason [1, Satz 5.6] shows that, for an arbitrary field E containing F, the kernel of the natural map $H^3E \to H^3(E(X))$ is trivial if ϕ_E is not a Pfister form and is $\{0, e_3(\phi_E)\}$ if ϕ_E is a Pfister form. Since the discriminant of ϕ is a_2, the form ϕ_E is a Pfister form if and only if $a_2 \in E^{\times 2}$. Therefore, the kernel of $H^3L_i \to H^3(L_i(X))$ is trivial for $i = 1, 3$ and is $\{0, (a_1, b, c)_{L_2}\}$ for $i = 2$. Now, $(a_1, b, c)_{L_2} = 0$ if and only if the Pfister form $\langle\langle a_1, b, c\rangle\rangle$ becomes isotropic over L_2; this condition is also equivalent to the existence of $b', c' \in F^\times$ such that

$$\langle\langle a_1, b, c\rangle\rangle = \langle\langle a_2, b', c'\rangle\rangle,$$

hence to ϕ being isotropic, as we noticed before. This proves the second part of the lemma.

A chase around the diagram above shows that β^2 induces a natural map from A to $\operatorname{Ker} v_3$ whose kernel is $\operatorname{Ker}(h(F) \to h(F(X)))$. To complete the proof, it thus suffices to show that this map is onto. This is clear if ϕ is isotropic, since then $\operatorname{Ker} v_3 = 0$. If ϕ is anisotropic, then we have seen above that

$$\operatorname{Ker} v_3 = \{0, (0, (a_1, b, c)_{L_2}, 0)\}.$$

Consider then $\sigma_3 \otimes (a_1, b, c) \in G \otimes H^3F$. From equation (21) and lemma 7, it follows that there exists $u \in H^2(F(X))$ such that

$$(a_1, b, c)_{F(X)} = (a_1) \cdot u = (a_2) \cdot u.$$

These relations readily yield: $v_2(\sigma_3 \otimes (a_1, b, c)) = \gamma_X^2(u)$. Moreover,

$$\beta^2(\sigma_3 \otimes (a_1, b, c)) = (0, (a_1, b, c)_{L_2}, 0),$$

hence $\sigma_3 \otimes (a_1, b, c)$ represents an element in A which is mapped to the non-trivial element of $\operatorname{Ker} v_3$. □

To complete the proof of proposition 5, it now suffices to show that A injects into a group which is trivial if ϕ is isotropic and of order 2 if ϕ is anisotropic, since the preceding lemma then implies $\operatorname{Ker}(h(F) \to h(F(X))) = 0$.

For $i = 1, 2$, let X^i denote the set of points of codimension i on X. We denote by $\partial_0 : H^2F(X) \to \oplus_{x \in X^1} H^1F(x)$ and $\partial_1 : \oplus_{x \in X^1} H^1F(x) \to \oplus_{y \in X^2} H^0F(y)$ the tame maps (see [3]), by $CH^2(X)$ the second Chow group of X and by $ch^2(X)$ the factor group:

$$ch^2(X) = CH^2(X)/2CH^2(X).$$

LEMMA 9. *There is a natural zero-sequence*:

$$0 \longrightarrow H^2F \longrightarrow H^2F(X) \xrightarrow{\partial_0} \bigoplus_{x \in X^1} H^1F(x) \xrightarrow{\partial_1} \bigoplus_{y \in X^2} H^0F(y) \longrightarrow ch^2(X) \longrightarrow 0$$

which is exact at every place except at $\oplus_{x \in X^1} H^1F(x)$. *If* ϕ *is not an anisotropic Pfister form, then the homology at this place is isomorphic to* H^1F.

Proof. Consider the following diagram, where the columns are exact and the rows are zero-sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & H^2F(X) & \xrightarrow{\partial_0} & \bigoplus_{x \in X^1} H^1F(x) & \xrightarrow{\partial_1} & \bigoplus_{y \in X^2} H^0F(y) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle R_2} & & \uparrow{\scriptstyle R_1} & & \uparrow{\scriptstyle R_0} & & \\
0 & \longrightarrow & K_2F(X) & \xrightarrow{\partial_0} & \bigoplus_{x \in X^1} K_1F(x) & \xrightarrow{\partial_1} & \bigoplus_{y \in X^2} K_0F(y) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle 2} & & \uparrow{\scriptstyle 2} & & \uparrow{\scriptstyle 2} & & \\
0 & \longrightarrow & K_2F(X) & \xrightarrow{\partial_0} & \bigoplus_{x \in X^1} K_1F(x) & \xrightarrow{\partial_1} & \bigoplus_{y \in X^2} K_0F(y) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \bigoplus_{x \in X^1} \mu_2(F(x)) & & 0 & &
\end{array}$$

where $\mu_2(F(x)) = \{\pm 1\} \subset F(x)^\times$. The homology groups of the second (and third) row are known from the papers [20] and [9] of Suslin and Karpenko: the homology at $K_2(F(X))$ is the isomorphic image of K_2F [20, Theorem 3.6, Cor. 5.6], the homology at $\oplus_{x \in X^1} K_1F(x)$ is isomorphic to K_1F under the map: $a \in K_1F \mapsto a \cdot s$ where s is a hyperplane section [9, Theorem 4.1], and the homology at $\oplus_{y \in X^2} K_0F(y)$ is $CH^2(X)$. It follows in particular that every element $\xi \in \oplus_{x \in X^1} K_1F(x)$ such that $2\xi \in \mathrm{Im}\,(\partial_0)$ can be represented, modulo $\mathrm{Im}\,(\partial_0)$, by an element in $\oplus_{x \in X^1} \mu_2(F(x))$.

A chase around the preceding diagram then shows that the homology of the second row at $K_2F(X)$ is mapped onto the homology of the top row at $H^2F(X)$, hence

$$\mathrm{Ker}\,\partial_0 = \mathrm{Coker}\,(K_2F \xrightarrow{2} K_2F) = H^2F.$$

This shows that the sequence of the lemma is exact at the first two places.

The diagram also yields a long homology sequence:

$$K_1F \xrightarrow{2} K_1F \longrightarrow \hom\Bigl(\bigoplus_{x\in X^1} H^1F(x)\Bigr)$$
$$\longrightarrow CH^2(X) \xrightarrow{2} CH^2(X) \longrightarrow \hom\Bigl(\bigoplus_{y\in X^2} H^0F(y)\Bigr) \longrightarrow 0$$

from which it readily follows that

$$\hom\Bigl(\bigoplus_{y\in X^2} H^0F(y)\Bigr) = \operatorname{Coker}\,(CH^2(X) \xrightarrow{2} CH^2(X)) = ch^2(X).$$

Moreover, if ϕ is not an anisotropic Pfister form, then Karpenko has shown in [9, Theorem 6.1] that $CH^2(X)$ has no torsion, hence

$$\hom\Bigl(\bigoplus_{x\in X^1} H^1F(x)\Bigr) = \operatorname{Coker}\,(K_1F \xrightarrow{2} K_1F) = H^1F. \qquad \square$$

Consider now the following commutative diagram, whose rows are zero-sequences:

$$\begin{array}{ccccccccccccc}
 & & 0 & & & & & & & & & & \\
 & & \uparrow & & & & & & & & & & \\
0\to A\to & & \operatorname{Coker}\gamma^2 & \longrightarrow & \operatorname{Coker}\gamma^2_X & & & & & & & & \\
 & & \uparrow & & \uparrow & & & & & & & & \\
0\to & & G\otimes H^3F & \to & G\otimes H^3F(X) & \to & \bigoplus_1 G\otimes H^2F(x) & & & & & & \\
 & & \uparrow{\scriptstyle\gamma^2} & & \uparrow{\scriptstyle\gamma^2_X} & & \uparrow{\scriptstyle\oplus\gamma^1_x} & & & & & & \\
 & & H^2F & \longrightarrow & H^2F(X) & \longrightarrow & \bigoplus_1 H^1F(x) & \longrightarrow & \bigoplus_2 H^0F(y) & & & & \\
 & & & & \uparrow{\scriptstyle\mathrm{cor}} & & \uparrow{\scriptstyle\oplus\,\mathrm{cor}} & & \uparrow{\scriptstyle\oplus\,\mathrm{cor}} & & & & \\
 & & & & H^2M(x) & \longrightarrow & \bigoplus_1 H^1M(x) & \longrightarrow & \bigoplus_2 H^0M(y) & \longrightarrow & ch^2X_M & \longrightarrow & 0 \\
 & & & & & & \uparrow{\scriptstyle\oplus\delta^1_x} & & \uparrow{\scriptstyle\oplus\delta^0_y} & & \uparrow & & \\
 & & & & & & \bigoplus_1\bigoplus_{i=1}^3 H^1L_i(x) & \to & \bigoplus_2\bigoplus_{i=1}^3 H^0L_i(y) & \to & \bigoplus_{i=1}^3 ch^2X_{L_i} & \to & 0 \\
 & & & & & & & & \uparrow{\scriptstyle\oplus\varepsilon^0_y} & & \uparrow & & \\
 & & & & & & & & \bigoplus_2 B(y) & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

where $B(y) = (G \otimes H^0F(y)) \oplus H^0M(y)$ and $C = (G \otimes ch^2X) \oplus ch^2X_M$. (For $E \supseteq F$ and $x \in X^i$, we denote: $E(x) = E \otimes_F F(x)$; thus $E(x)$ is not necessarily a field, but $\bigoplus_{x \in X^i} H^jE(x) = \bigoplus_{z \in X^i_E} H^jE(z)$).

All the columns, except possibly the last one, are exact: this follows from propositions 1 and 4 for the second column and from exactness of S^1 and S^0 (even in the case of étale algebras instead of fields) for the third and fourth columns. A diagram chase in the spirit of the snake lemma then yields a map from A to the homology group of the last column. Moreover, the first row is exact at A (by definition of A), the second at $G \otimes H^3F$ by a theorem of Arason [1, Satz 5.6], the third, the fifth and the sixth at $H^2F(X)$, $\bigoplus_2 \bigoplus_i H^0L_i(y)$ and $G \otimes ch^2X$ respectively, by the preceding lemma. Since ϕ_M is isotropic (and even hyperbolic), the same lemma shows that the homology of the fourth (resp. third) row at $\bigoplus_1 H^1M(x)$ (resp. $\bigoplus_1 H^1F(x)$) is isomorphic to H^1M (resp. H^1F). Since S^1 is exact, it follows that every element of the homology group at $\bigoplus_1 H^1M(x)$ which becomes trivial in the homology group at $\bigoplus_1 H^1F(x)$ can be represented by the image of an element in $\bigoplus_1 \bigoplus_{i=1}^3 H^1L_i(x)$. Therefore, another diagram chase shows that the map from A to the homology group of the last column is injective.

To complete the proof, it now suffices to show that the latter homology group is trivial if ϕ is isotropic and of order 2 if ϕ is anisotropic. This follows from Karpenko's computation of Chow groups [9]: for any field E containing F, if ϕ_E is an anisotropic Pfister form, then $CH^2X_E \simeq \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; otherwise $CH^2X_E \simeq \mathbb{Z}$. Now, ϕ_M is isotropic and ϕ_{L_1}, ϕ_{L_3} are not Pfister forms since a_2 is not a square in L_1 nor L_3, hence

$$ch^2X_M \simeq ch^2X \simeq ch^2X_{L_i} \simeq \mathbb{Z}/2\mathbb{Z} \qquad \text{for } i = 1, 3.$$

On the other hand, ϕ_{L_2} is a Pfister form. If it is isotropic, then it is hyperbolic, hence $\langle\langle a_1, b, c\rangle\rangle_{L_2} = 0$. It then follows that

$$\langle\langle a_1, b, c\rangle\rangle = \langle\langle a_2, b', c'\rangle\rangle$$

for some $b', c' \in F^\times$, hence ϕ is isotropic. Thus,

$$ch^2X_{L_2} \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } \phi \text{ is isotropic} \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } \phi \text{ is anisotropic.} \end{cases}$$

The conclusion now readily follows from the description of the maps induced on Chow groups modulo 2 by $\oplus\, \varepsilon_y^0$ and $\oplus\, \delta_y^0$. $\square$

Added in proof: Substantial parts of Theorems A and B have been proved by different methods by Bruno Kahn in his thesis *Représentations galoisiennes et classes caractéristiques* (Univ. Paris VII, 1987) (see the chapter *Divisibilité du groupe de Brauer*).

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