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## Density of states in spectral geometry

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### Introduction

Recent studies on spectral geometry threw a light on the relationships between a discontinuous action of a group and the spectrum of the Laplacian (or more generally the spectra of elliptic operators) on a non-compact Riemannian manifold. The first result in this direction is the observation by R. Brooks [B] that the bottom of the  $L^2$ -spectrum of the Laplacian is related to the *amenability* of discontinuous transformation groups (see also [KOS] and [S1]). The purpose of this paper is to investigate the *integrated density of states* of a *periodic* Schrödinger operator on a manifold with *compact quotient* from the same point of view.

The integrated density of states, which is the concept introduced first by physicists in quantum theory of solids, is a non-decreasing function  $\varphi(\lambda)$  on the real line defined roughly as the number of possible energy levels in the interval  $(-\infty, \lambda)$  divided by the volume of a sufficiently large domain. To justify this physical definition, we must impose a suitable boundary condition on eigenfunctions and specify the way how to blow up the domain filling the whole space. For the Schrödinger operator with a *periodic potential* on the Euclidean space, a classical observation (cf. [Sh]) says that  $\varphi(\lambda)$  is well-defined as far as the domain blows up in a sufficiently regular way and does not depend on the choice of the boundary conditions; say, Dirichlet, Neumann, and periodic boundary conditions. It is also a fact that the spectrum of the Schrödinger operator on the whole space is characterized as the set of increasing points of  $\varphi(\lambda)$ . One of the results in this paper gives a partial generalization of those facts to the case of a Riemannian manifold with compact quotient. In the discussion, we shall see a prominent role of amenability of discontinuous groups acting on manifolds, together with the role of spectral distribution functions defined by means of the concept of the *von Neumann trace*. See [SN], [S3], and [KOS] for the general background for the spectral theory of periodic Schrödinger operators on a manifold, also [ES] which gives us the stimulus for writing this paper.

## §1. Definitions and results

Let  $X$  be a complete, connected, noncompact Riemannian manifold of dimension  $n$ , and  $\mathcal{D} = \{D_j\}_{j=1}^{\infty}$  be a family of bounded connected open sets in  $X$  with *piecewise smooth* boundaries satisfying

$$\bar{D}_j \subset D_{j+1}, \quad \bigcup_{j=1}^{\infty} D_j = X.$$

Let  $q$  be a smooth real-valued function on  $X$ . Consider the Schrödinger operator  $H_{D_j} = -\Delta_{D_j} + q$  on each  $D_j$  acting on  $L^2(D_j)$  with Dirichlet boundary conditions. We denote by  $\varphi_{D_j}(\lambda)$  the number of eigenvalues of  $H_{D_j}$ , not exceeding  $\lambda$ , where each eigenvalue is repeated according to its multiplicity. We now define the function  $\varphi_{\mathcal{D}}$  by the limit (if it exists)

$$\varphi_{\mathcal{D}}(\lambda) = \lim_{j \rightarrow \infty} \text{vol}(D_j)^{-1} \varphi_{D_j}(\lambda),$$

and call  $\varphi_{\mathcal{D}}$  the integrand density of states for the Schrödinger operator  $H_X = -\Delta_X + q$  associated with the family  $\mathcal{D}$ . The questions with which we are concerned are: (1) Under what condition does the limit exist? (2) When it exists, is  $\varphi_{\mathcal{D}}$  independent of the choice of the expanding family  $\mathcal{D}$ ?

Given a manifold with *compact quotient*, we may introduce the integrated density of states associated with *periodic boundary conditions*. Here a complete Riemannian manifold  $X$  is said to have compact quotient if there is a discrete subgroup  $\Gamma$  in the isometry group of  $X$  acting discontinuously on  $X$  such that the quotient space  $M = \Gamma \backslash X$  is compact. We assume that  $q$  is  $\Gamma$ -invariant so that  $q$  may be regarded as a function on  $M$ . Let

$$H_X = \int \lambda \, dE(\lambda)$$

denote the spectral resolution of  $H_X$ . We define the *spectral distribution function*  $\Phi_{\Gamma}$  by

$$\Phi_{\Gamma}(\lambda) = \text{Tr}_{\Gamma} E(\lambda),$$

where  $\text{Tr}_{\Gamma}$  is the standard *von Neumann trace* on the von Neumann algebra of  $\Gamma$ -equivariant bounded operators of  $L^2(X)$  (see [At], [ES], [S3]).

From the definition, it follows easily that the quantity  $\text{vol}(\Gamma \backslash X)^{-1} \Phi_{\Gamma}(\lambda)$  depends only on the commensurability class of  $\Gamma$ ; that is, if  $\Gamma_1$  and  $\Gamma_2$  are

discontinuous transformation groups of  $X$  such that  $q$  is invariant under  $\Gamma_1$  and  $\Gamma_2$ , and  $\Gamma_1 \cap \Gamma_2$  is of finite index in both  $\Gamma_1$  and  $\Gamma_2$ , then one has

$$\text{vol}(\Gamma_1 \backslash X)^{-1} \Phi_{\Gamma_1} = \text{vol}(\Gamma_2 \backslash X)^{-1} \Phi_{\Gamma_2}.$$

In the special case that  $q \equiv 0$  and  $X$  is a homogeneous Riemannian manifold, the quantity  $\text{vol}(\Gamma \backslash X)^{-1} \Phi_{\Gamma}(\lambda)$  does not depend on  $\Gamma$ . For example, if  $X = \mathbb{R}^2$ , one has

$$\text{vol}(\Gamma \backslash X)^{-1} \Phi_{\Gamma}(\lambda) = (4\pi)^{-1} \lambda, \quad \lambda \geq 0,$$

and if  $X = \mathbb{H}^2$ , the hyperbolic 2-plane, one has

$$\text{vol}(\Gamma \backslash X)^{-1} \Phi_{\Gamma}(\lambda) = (4\pi)^{-1} \int_0^{\lambda - 1/4} \tanh \pi \sqrt{\lambda} \, d\lambda, \quad \lambda \geq 1/4.$$

To see that the function  $\Phi_{\Gamma}$  may be regarded as the integrated density of states associated with periodic boundary value conditions, we suppose that  $\Gamma$  acts freely on  $X$  and has a family of normal subgroup  $\{\Gamma_i\}_{i=1}^{\infty}$  such that  $\Gamma_i$  is of finite index in  $\Gamma$ ,  $\Gamma_{i+1}$  is contained in  $\Gamma_i$ , and  $\bigcap_{i=1}^{\infty} \Gamma_i = \{1\}$ . We then have a tower of finite-fold covering maps of closed manifolds

$$\cdots \longrightarrow M_{i+1} \longrightarrow M_i = \Gamma_i \backslash X \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M.$$

Let  $\Phi_{M_i}(\lambda)$  denote the number of eigenvalues of  $H_{M_i}$  on the closed manifold  $M_i$  not exceeding  $\lambda$ . In [SN], it was observed that

$$\text{vol}(\Gamma \backslash X)^{-1} \Phi_{\Gamma}(\lambda) = \lim_{i \rightarrow \infty} \text{vol}(M_i)^{-1} \Phi_{M_i}(\lambda)$$

at all points of continuity of  $\Phi_{\Gamma}$ . It should be noted ([ES], [SN]) that the set of increasing points of  $\Phi_{\Gamma}$  coincides with the spectrum of  $H_X$ .

It is natural to compare  $\varphi_{\mathcal{D}}(\lambda)$  with  $\Phi_{\Gamma}(\lambda)$ . In the case that  $X$  is the Euclidean space  $\mathbb{R}^n$  and  $\mathcal{D}$  is a family of concentric balls, it is known that  $\varphi_{\mathcal{D}}$  exists and coincides with  $\text{vol}(\Gamma \backslash X)^{-1} \Phi_{\Gamma}$ . On the other hand, if  $X = \mathbb{H}^n$ , the  $n$ -dimensional hyperbolic space, and  $\mathcal{D}$  is a family of concentric geodesic balls in  $\mathbb{H}^n$ , we observe that  $\varphi_{\mathcal{D}}$  is not equal to  $\text{vol}(\Gamma \backslash X)^{-1} \Phi_{\Gamma}$  (see Section 3). This is due to different geometric features of geodesic balls in  $\mathbb{R}^n$  and  $\mathbb{H}^n$  which may be clarified if one looks at the ratio

$$\text{vol}(\partial_h D_j) / \text{vol}(D_j), \quad h > 0,$$

where  $\partial_h D$  denotes the “thick” boundary  $\{x \in D; \text{dist}(x, \partial D) \leq h\}$ . In fact, for  $\mathbb{R}^n$ , this goes to zero as  $j \rightarrow \infty$  for every  $h$ , while, for  $\mathbb{H}^n$ , this goes to the positive number  $1 - e^{-h(n-1)}$ . In terms of discrete transformation groups, this corresponds to the fact that a group  $\Gamma$  acting discontinuously on  $\mathbb{R}^n$  is *amenable*, and a group  $\Gamma$  acting on  $\mathbb{H}^n$  is non-amenable. Indeed, we may prove the following general criterion of amenability.

**PROPOSITION 1.1.** *The transformation group  $\Gamma$  is amenable if and only if there exists an expanding family  $\mathcal{D} = \{D_j\}$  of bounded domains with piecewise smooth boundaries satisfying the following property:*

$$\lim_{j \rightarrow \infty} \text{vol}(\partial_h D_j) / \text{vol}(D_j) = 0 \quad (\text{P})$$

for every  $h > 0$ .

In light of this criterion, we now state the main theorem of this paper, a generalization of the classical result for  $X = \mathbb{R}^n$ .

**THEOREM 1.1.** *If an expanding family  $\mathcal{D} = \{D_j\}$  satisfies the property (P) in the above proposition, then  $\varphi_{\mathcal{D}}$  exists and equals  $\text{vol}(\Gamma \backslash X)^{-1} \Phi_{\Gamma}$  at all points of continuity of  $\Phi_{\Gamma}$ .*

An immediate consequence of this theorem is that, if  $\Gamma$  is amenable, the integrated density of states  $\varphi_{\mathcal{D}}$  does not depend on the expanding family  $\mathcal{D}$  with the property (P). We also conclude that  $\text{vol}(\Gamma \backslash X)^{-1} \Phi_{\Gamma}$  does not depend on  $\Gamma$ , which is by no means trivial from the definition of  $\Phi_{\Gamma}$  since  $X$  is not supposed to be homogeneous.

It is interesting to consider the density of states associated with *Neumann boundary conditions*. We conjecture that the same statements as in Theorem 1 hold. Sznitman [Sz2] shows that, for the hyperbolic space, the integrated density of states associated with Dirichlet boundary conditions is different from that associated with Neumann boundary conditions.

## §2. Families of expanding domains and limit relations for the heat kernels

Henceforth we assume that  $X$  is a Riemannian manifold with compact quotient  $\Gamma \backslash X$ . We choose a fundamental domain  $\mathcal{F}$  for the action of  $\Gamma$  with compact closure. The distance function on  $X$  will be denoted by  $d(x, y)$ .

Let  $k(t, x, y)$  denote the heat kernel function for the semi-group  $\exp(-tH_X)$ , and  $k_D(t, x, y)$  the heat kernel function on a domain  $D$  associated with Dirichlet boundary conditions. We readily get

$$\int e^{-\lambda t} d\varphi_D(\lambda) = \int_D k_D(t, x, x) dx.$$

The following lemma on the spectral distribution function  $\Phi_\Gamma$  is immediate from the definition of  $\Gamma$ -trace.

$$\text{LEMMA 2.1. } \int e^{-\lambda t} d\Phi_\Gamma(\lambda) = \int_{\mathcal{F}} k(t, x, x) dx.$$

The idea of proof of Theorem 1.1 is based on a uniform estimate of the difference between the diagonal of the heat kernel and that of the Dirichlet heat kernel.

LEMMA 2.2. *Given a positive  $T$ , we have positive constants  $C_1$  and  $C_2$  such that*

$$0 \leq k(t, x, y) \leq C_1 t^{-n/2} \exp(-C_2 d(x, y)^2/t) \quad (1)$$

for  $t \in (0, T]$ , and

$$0 \leq k(t, x, y) - k_D(t, x, y) \leq C_1 t^{-n/2} \exp(-C_2 d(y, \partial D)^2/t) \quad (2)$$

for  $0 < t \leq \min(T, 2C_2 d(y, \partial D)^2/n)$ .

*Proof.* The first inequality (1) is due to [Do] (see also [BS]). The second inequality is a consequence of the maximum principle (see [C] and [D]).

PROPOSITION 2.1. *If the family  $\mathcal{D}$  satisfies the property (P) then*

$$\lim_{j \rightarrow \infty} \text{vol}(D_j)^{-1} \int_{D_j} (k(t, x, x) - k_{D_j}(t, x, x)) dx = 0.$$

*Proof.* Let  $t > 0$ , and take constants  $C_1$  and  $C_2$  in (1) for  $T = t$ . We have

$$\begin{aligned} & \text{vol}(D_j)^{-1} \int_{D_j} (k(t, x, x) - k_{D_j}(t, x, x)) dx \\ &= \text{vol}(D_j)^{-1} \int_{\partial_h D_j} (k(t, x, x) - k_{D_j}(t, x, x)) dx \\ & \quad + \text{vol}(D_j)^{-1} \int_{D_j \setminus \partial_h D_j} (k(t, x, x) - k_{D_j}(t, x, x)) dx. \end{aligned}$$

In view of Lemma 2.2, (1), the first term is estimated from above by

$$C_1 t^{-n/2} \text{vol}(\partial_h D_j) / \text{vol}(D_j),$$

which tends to zero as  $j \uparrow \infty$ . Take  $h$  with  $t \leq 2C_2 h^2/n$ . Then, for  $x \in D_j \setminus \partial_h D_j$ , one has  $t \leq 2C_2 d(x, \partial D_j)^2/n$ , so that, by Lemma 2.2, (2) the second term is estimated from above by

$$C_1 t^{-n/2} \exp(-C_2 h^2/t).$$

By letting  $h$  go to infinity, we get the assertion.

**PROPOSITION 2.2.** *If  $\mathcal{D}$  satisfies the property (P), then one has, for every  $\Gamma$ -periodic continuous function  $f$ , that*

$$\lim_{j \rightarrow \infty} \text{vol}(D_j)^{-1} \int_{D_j} f(x) dx = \text{vol}(\mathcal{F})^{-1} \int_{\mathcal{F}} f(x) dx.$$

*Proof.* Put  $E_j = \{\sigma \in \Gamma; (D_j \setminus \partial_h D_j) \cap \sigma \mathcal{F} \neq \emptyset\}$ , and

$$D'_j = \bigcup_{\sigma \in E_j} \sigma \mathcal{F}.$$

It is clear that  $(D_j \setminus \partial_h D_j) \subset D'_j$ . We show that, if  $h \geq \text{diam}(\mathcal{F})$ , then  $D'_j \subset D_j$ . Let  $x \in (D_j \setminus \partial_h D_j) \cap \sigma \mathcal{F}$ . since  $d(x, \partial D_j) \geq h$ , we find  $\partial D_j \cap B_h(x) = \emptyset$ , where  $B_h(x) = \{z \in X; d(x, z) \leq h\}$ . From the connectedness of  $B_h(x)$ , it follows that  $B_h(x) \subset D_j$ . Since  $h \geq \text{diam}(\mathcal{F})$ , we have  $\sigma \mathcal{F} \subset B_h(x) \subset D_j$ .

We now find

$$\begin{aligned} \text{vol}(D_j)^{-1} \int_{D_j} f(x) dx &= \text{vol}(D_j)^{-1} \int_{D_j} f(x) dx + \text{vol}(D_j)^{-1} \int_{D_j \setminus D'_j} f(x) dx \\ &= \frac{\text{vol}(D'_j)}{\text{vol}(D_j)} \frac{1}{\text{vol}(D'_j)} \int_{D'_j} f(x) dx \\ &\quad + \frac{\text{vol}(D_j \setminus D'_j)}{\text{vol}(D_j)} \frac{1}{\text{vol}(D_j \setminus D'_j)} \int_{D_j \setminus D'_j} f(x) dx. \end{aligned}$$

Since  $D_j \setminus D'_j \subset \partial_h D_j$ , we have  $\lim_{j \rightarrow \infty} \text{vol}(D_j \setminus D'_j) / \text{vol}(D_j) = 0$  and  $\lim_{j \rightarrow \infty} \text{vol}(D'_j) / \text{vol}(D_j) = 1$ . In view of the  $\Gamma$ -periodicity of  $f$ , we find

$$\begin{aligned} \lim_{j \rightarrow \infty} \text{vol}(D_j)^{-1} \int_{D_j} f(x) d(x) &= \lim_{j \rightarrow \infty} \text{vol}(D'_j)^{-1} \int_{D'_j} f(x) dx \\ &= \text{vol}(\mathcal{F})^{-1} \int_{\mathcal{F}} f(x) dx. \end{aligned}$$

We shall make use of the following genral lemma to complete the proof of Theorem 1.1.

LEMMA 2.3 (cf. [Sh]). *Let  $\{\varphi_j(\lambda)\}_{j=1}^{\infty}$  be a sequence of non-decreasing functions with  $\varphi_j(\lambda) = 0$  for  $\lambda \leq c$ , where  $c$  is a constant not depending on  $j$ . Suppose that there exists a function  $C(t)$ , not depending on  $j$  such that*

$$\Phi_j(t) := \int e^{-\lambda t} d\varphi_j(t) \leq C(t),$$

and

$$\lim_{j \rightarrow \infty} \Phi_j(t) = \int e^{-\lambda t} d\varphi(\lambda),$$

where  $\varphi$  is a non-decreasing function. Then  $\lim_{j \rightarrow \infty} \varphi_j(\lambda) = \varphi(\lambda)$  at all points of continuity of  $\varphi(\lambda)$ .

We apply this lemma to

$$\varphi_j(\lambda) = \text{vol}(D_j)^{-1} \varphi_{D_j}(\lambda),$$

$$\varphi(\lambda) = \text{vol}(\Gamma \setminus X)^{-1} \Phi_{\Gamma}(\lambda).$$

Since the first eigenvalue of  $H_{D_j}$  is not less than  $\min q(x)$ , we observe that  $\varphi_j(\lambda) = 0$  for  $\lambda < \min q(x)$ . We also find that

$$\begin{aligned} \Phi_j(\lambda) &= \text{vol}(D_j)^{-1} \int_{D_j} k_{D_j}(t, x, x) dx \\ &\leq \text{vol}(D_j)^{-1} \int_{D_j} k(t, x, x) dx \\ &\leq \sup_{x \in X} k(t, x, x) =: C(t), \end{aligned}$$



where we should note that the function  $k(t, x, x)$  is  $\Gamma$ -periodic with respect to the variable  $x$ . By Proposition 2.1,

$$\begin{aligned} \lim_{j \rightarrow \infty} \Phi_j(t) &= \lim_{j \rightarrow \infty} \text{vol}(D_j)^{-1} \int_{D_j} k_{D_j}(t, x, x) dx \\ &= \lim_{j \rightarrow \infty} \text{vol}(D_j)^{-1} \int_{D_j} k(t, x, x) dx. \end{aligned}$$

Using again  $\Gamma$ -periodicity of  $k(t, x, x)$ , together with Proposition 2.2, we have

$$\begin{aligned} \lim_{j \rightarrow \infty} \Phi_j(t) &= \text{vol}(\mathcal{F})^{-1} \int_{\mathcal{F}} k(t, x, x) dx \\ &= \text{vol}(\mathcal{F})^{-1} \int e^{-\lambda t} d\Phi_{\Gamma}(\lambda) \\ &= \int e^{-\lambda t} d\varphi(\lambda) \end{aligned}$$

as desired.

### §3. Manifolds with amenable group actions

In this section, we shall prove Proposition 1.1 in a slightly strong form. For this, we recall the Følner’s characterization of amenability. Let  $\Gamma$  be a finitely generated group with a fixed finite set  $A$  of generators.

**PROPOSITION 3.1** (Følner [F] and [Ad]).  *$\Gamma$  is amenable if and only if, for every positive  $\varepsilon$ , there exists a non-empty finite set  $E$  such that*

$$|EA \setminus E| \leq \varepsilon |E|,$$

where  $|E|$  denotes the cardinality of the set  $E$ .

We first assume that a manifold  $X$  with compact quotient  $\Gamma \backslash X$  has a family  $\{D_j\}$  satisfying the property (P). Fixing a fundamental domain  $\mathcal{F}$ , we put

$$A = \{a \in \Gamma : a\mathcal{F} \cap \mathcal{F} \neq \emptyset\}.$$

The finite set  $A$  generates  $\Gamma$ . Taking a number  $h > 2 \cdot \text{diam}(\mathcal{F})$ , we set

$$E_j = \{\gamma \in \Gamma; \gamma\bar{\mathcal{F}} \cap (D_j \setminus \partial_h D_j) \neq \emptyset\}.$$

Let  $\sigma = \gamma \cdot a \in E_j A$  ( $\gamma \in E_j, a \in A$ ). We shall prove that  $\sigma\bar{\mathcal{F}} \subset D_j$ . For this, let  $z \in \gamma\bar{\mathcal{F}} \cap (D_j \setminus \partial_h D_j)$ . We then have  $B_h(z) \subset D_j$  as before. Since

$$\sigma\bar{\mathcal{F}} \cap \gamma\bar{\mathcal{F}} = \gamma(a\bar{\mathcal{F}} \cap \bar{\mathcal{F}}) \neq \emptyset,$$

there exists an element  $y \in \sigma\bar{\mathcal{F}} \cap \gamma\bar{\mathcal{F}}$ , and hence, for every  $x \in \sigma\bar{\mathcal{F}}$ , one has

$$d(x, z) \leq d(x, y) + d(y, z) \leq 2 \cdot \text{diam}(\mathcal{F}) < h,$$

which implies that  $\sigma\bar{\mathcal{F}} \subset B_h(z)$ , and hence  $\sigma\bar{\mathcal{F}} \subset D_j$ .

We now observe

$$\begin{aligned} |E_j A \setminus E_j|/|E_j| &= \frac{|E_j A| \text{vol}(\mathcal{F})}{|E_j| \text{vol}(\mathcal{F})} - 1 \leq \frac{\text{vol}(D_j)}{\text{vol}(D_j \setminus \partial_h D_j)} - 1 \\ &= \frac{\text{vol}(\partial_h D_j)}{\text{vol}(D_j)} \left(1 - \frac{\text{vol}(\partial_h D_j)}{\text{vol}(D_j)}\right)^{-1}, \end{aligned}$$

which goes to zero as  $j \uparrow \infty$ . Hence  $\Gamma$  is amenable by Følner's criterion.

Next we suppose that  $\Gamma$  is amenable. Using a smooth triangulation of the orbifold  $\Gamma \backslash X$ , we may lift up  $n$ -simplices one by one to  $X$  to obtain a *connected polyhedral* fundamental domain  $\mathcal{F}$ . The finite set  $A = \{\sigma \in \Gamma; \sigma\bar{\mathcal{F}} \cap \bar{\mathcal{F}} \neq \emptyset\}$  is symmetric and contains the unit element. We associate the *Cayley graph*  $\mathcal{C}(\Gamma, A)$ ; the set of vertices being  $\Gamma$  and the set of edges being  $\{(\gamma, \sigma) \in \Gamma \times \Gamma; \gamma^{-1}\sigma \in A\}$ . We denote by  $d_A$  the distance function on  $\Gamma$  associated with the graph  $\mathcal{C}(\Gamma, A)$ . A subset  $E$  in  $\Gamma$  will be called *connected* if, for any two vertices in  $E$ , there exists a path in  $\mathcal{C}(\Gamma, A)$  joining those vertices and consisting of vertices in  $E$ . By use of Theorem 4 in [Ad], there is a family  $\{E_j\}_{j=1}^\infty$  of connected subsets of  $\Gamma$  such that

$$\begin{aligned} \bigcup_{j=1}^\infty E_j &= \Gamma, \quad E_j \subset E_j \cdot A \subset E_{j+1} \quad \text{and} \\ |E_j \cdot A^j \setminus E_j| &\leq |E_j|/j|A|^j \quad \text{for every } j. \end{aligned}$$

We put  $F_j = \bigcup_{\gamma \in E_j} \gamma\bar{\mathcal{F}}$  and  $F'_j = \bigcup_{\gamma \in E_j \cdot A} \gamma\bar{\mathcal{F}}$ , which are connected by the choice of  $A$  and the connectedness of  $E_j$ . It should be noted that there exists a positive  $\varepsilon$  such that the  $\varepsilon$ -neighborhood of  $F_j$  is contained in  $F'_j$ . Thus we may make a uniform

regularization  $D_j$  of  $F_j$  satisfying  $F_j \subset D_j \subset \bar{D}_j \subset F'_j$  (see [B]). It is clear that  $\bigcup_{j=1}^{\infty} D_j = X$  and  $\bar{D}_j \subset D_{j+1}$ . Our goal is to show that  $\{D_j\}_{j=1}^{\infty}$  satisfies the property (P). Let  $x_0 \in \mathcal{F}$ . Since the map  $f: \Gamma \rightarrow X$ ,  $f(\gamma) = \gamma x_0$ , is a rough isometry (Kanai [K]), we have

$$d_A(\gamma, \mu) \leq c_1 d(\gamma x_0, \mu x_0) + c_2$$

with suitable constants  $c_1 > 0$  and  $c_2 \geq 0$ .

LEMMA 3.1. *If  $h \leq (j - c_2)/c_1 - 2 \cdot \text{diam}(\mathcal{F})$ , then the thick boundary  $\partial_h D_j$  is contained in the set*

$$\partial^j F'_j = \bigcup \{ \mu \sigma \mathcal{F}; \sigma \in A, \mu \in E_j, \text{ and there is } \gamma \in A^j \text{ with } \mu \gamma \notin E_j \}.$$

*Proof.* Suppose  $x$  is contained in  $\partial_h D_j \cap \mu \bar{\mathcal{F}}$  for some  $\mu \in E_j$ . Since  $F_j \subset D_j$  there is  $y \in \bar{X} \setminus \bar{F}_j$  with  $d(x, y) \leq h$ . Choose  $\rho \notin E_j$  so that  $y \in \rho \bar{\mathcal{F}}$ . Then  $d(\mu x_0, \rho x_0) \leq h + 2 \cdot \text{diam}(\mathcal{F})$ , hence  $d_A(\mu, \rho) \leq j$  and  $\partial_h D_j \cap F_j \subset \partial^j F_j$ , where

$$\partial^j F_j = \bigcup \{ \mu \mathcal{F} \mid \mu \in E_j \text{ and there is } \gamma \in A^j \text{ with } \mu \gamma \notin E_j \}.$$

If  $\gamma \in E_j A \setminus E_j$ , it is clear that  $\gamma \mathcal{F} \subset \partial^j F'_j$  (since  $A \subset A^j$ ), therefore

$$\partial_h D_j \subset (\partial_h D_j \cap F_j) \cup (F'_j \setminus F_j) \subset \partial^j F'_j.$$

We now show that the family  $\{D_j\}_{j=1}^{\infty}$  satisfies the property (P). By the definition of  $\partial^j F_j$  and  $\partial^j F'_j$  we have

$$\begin{aligned} \text{vol}(\partial^j F'_j) &\leq |A| \cdot \text{vol}(\partial^j F_j) \\ &= |A| \cdot \text{vol}(\mathcal{F}) \cdot |\{ \mu \in E_j \mid \text{there is } \gamma \in A^j \text{ with } \mu \gamma \notin E_j \}| \\ &\leq |A| \cdot \text{vol}(\mathcal{F}) \sum_{\gamma \in A^j} |E_j \setminus E_j \gamma^{-1}| \\ &= |A| \cdot \text{vol}(\mathcal{F}) \sum_{\gamma \in A^j} |E_j \gamma \setminus E_j| \\ &\leq |A| \cdot \text{vol}(\mathcal{F}) \cdot |A^j| \cdot |E_j A^j \setminus E_j| \\ &\leq \text{vol}(\mathcal{F}) \cdot |E_j|/j \\ &= \text{vol}(F_j)/j. \end{aligned}$$

Therefore we get, for every  $h > 0$ , that

$$\text{vol}(\partial_h D_j)/\text{vol}(D_j) \leq \text{vol}(\partial^j F_j)/\text{vol}(F_j) \leq 1/j \rightarrow 0.$$

Summarizing up, we obtain

**PROPOSITION 3.2.** *If  $\Gamma$  is amenable, then there exists an expanding family  $\mathcal{D} = \{D_j\}$  of bounded open domains with smooth boundaries satisfying the following conditions:*

- (1)  $\mathcal{D}$  has the property (P),
- (2) the boundary  $\partial D_j$  has a uniformly bounded second fundamental form  $h_j$ .  
More precisely, there exists positive constant  $c$  not depending on  $j$  with  $-cg \leq h_j \leq cg$ , where  $g$  denotes the Riemannian metric on  $X$ .

A group of subexponential growth is amenable (see [B]). In this case, we may construct a family  $\mathcal{D} = \{D_j\}$  satisfying the conditions in the above proposition by using the following property on concentric geodesic balls.

**LEMMA 3.2.** *Suppose that  $\Gamma$  is of subexponential growth. For an arbitrary point  $x$  in  $X$ , there is a sequences of positive numbers  $\{R_j\}_{j=1}^\infty$  such that*

- (1)  $R_j \uparrow \infty$ ,
- (2)  $\lim_{j \rightarrow \infty} \text{vol}(B_{R_j}(x))/\text{vol}(B_{R_j-h}(x)) = 1$  for every  $h > 0$ .

(cf. [Ad]).

#### §4. Hyperbolic spaces

We now consider the density of states associated with the Laplacian on the hyperbolic space  $X = \mathbb{H}^n$ . The manifold  $\mathbb{H}^n$  is a typical example of a manifold with a non-amenable discontinuous transformation group.

**THEOREM 4.1.** *Let  $\mathcal{D} = \{D_j\}$  be a family of concentric geodesic balls in  $\mathbb{H}^n$ . Then one has*

$$\text{vol}(\mathcal{F})^{-1} \int e^{-\lambda t} d\Phi_\Gamma(\lambda) > \limsup_{j \rightarrow \infty} \text{vol}(D_j)^{-1} \int e^{-\lambda t} d\varphi_{D_j}(\lambda).$$

In particular,  $\text{vol}(\Gamma \backslash X)^{-1} \Phi_\Gamma \neq \varphi_{\mathcal{D}}$ .

*Proof.* Since  $\mathbb{H}^n$  is a homogeneous Riemannian manifold,  $k(t, x, x)$  does not depend on the variable  $x$ , so that we write

$$k(t) = k(t, x, x).$$

We then find

$$\begin{aligned} & \text{vol}(\mathcal{F})^{-1} \int e^{-\lambda t} d\Phi_{\Gamma}(\lambda) - \text{vol}(D_j)^{-1} \int_{D_j} k_{D_j}(t, x, x) dx \\ &= k(t) - \text{vol}(D_j)^{-1} \int_{D_j} k_{D_j}(t, x, x) dx \\ &= \text{vol}(D_j)^{-1} \int_{D_j} (k(t) - k_{D_j}(t, x, x)) dx \\ &\geq \text{vol}(D_j)^{-1} \int_{\partial_h D_j} (k(t) - k_{D_j}(t, x, x)) dx, \end{aligned}$$

where we have used the fact that  $k_D(t, x, y) \leq k(t)$ .

To complete the proof, we need the following lemma.

**LEMMA 4.1.** *For a fixed  $t > 0$ , there exists a positive  $h$  such that*

$$k_D(t, x, x) \leq k(t)/2$$

for every geodesic ball  $D$  and every  $x \in \partial_h D$ .

*Proof.* Choose a unit speed geodesic  $C : \mathbb{R} \rightarrow X$ , and consider the horoball  $H = \bigcup_{\tau > 0} B_{\tau}(c(\tau))$ . Let  $k_H(t, x, y)$  denote the Dirichlet heat kernel function for the horoball. Since  $\lim_{x \rightarrow \partial H} k_H(t, x, x) = 0$ , it follows that there exists a positive  $h$  such that, for a positive  $\delta$  with  $\text{dist}(c(\delta), \partial H) = \delta \leq h$ .

$$k_H(t, c(\delta), c(\delta)) \leq k(t)/2.$$

Let  $x \in \partial_h D$ . Since one can find an isometry  $f$  on  $\mathbb{H}^n$  such that  $f(D) = B_{\tau}(c(\tau))$ ,  $\tau > 0$ , and  $f(x) = c(\delta)$  for some  $\delta \leq h$ . Hence we have, by the domain monotonicity of the Dirichlet heat kernel,

$$\begin{aligned} k_D(t, x, x) &= k_{B_{\tau}(c(\tau))}(t, c(\delta), c(\delta)) \\ &\leq k_H(t, c(\delta), c(\delta)) \leq k(t)/2, \end{aligned}$$

as desired.

Applying the above lemma, we get

$$\begin{aligned} \text{vol}(\mathcal{F})^{-1} \int e^{-\lambda t} d\Phi_r(\lambda) - \text{vol}(D_j)^{-1} \int_{D_j} k_{D_j}(t, x, x) dx \\ \geq \frac{k(t)}{2} \text{vol}(\partial_h D_j) / \text{vol}(D_j). \end{aligned}$$

If  $r_j$  denotes the radius of  $D_j$ , one has  $\text{vol}(D_j) = e^{(n-1)r_j}$ , so that the last term is written as

$$\frac{k(t)}{2} (1 - e^{-(n-1)h}) > 0.$$

This completes the proof of Theorem 4.1.

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