Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 68 (1993)

Artikel: Realization of simply-connected 4-manifolds with a given boundary.

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DOI: https://doi.org/10.5169/seals-51759

Comment. Math. Helvetici 68 (1993) 20–47
0010–2571/93/010020–28$1.50 + 0.20/0

Realization of simply-connected 4-manifolds with a given boundary

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Introduction

One of the nicest applications of Michael Freedman's work in 4-dimensional topology is his classification of closed, simply-connected, oriented 4-manifolds in terms of their intersection pairings and their Kirby–Siebenmann invariants (see §10.1 of [FQ]). A similar classification for compact, simply-connected, oriented 4-manifolds with connected boundary was begun in [B2]. The new features which arose were the relationship between the intersection pairing of the 4-manifold and

(i) the torsion link pairing of the bounding 3-manifolds;
(ii) the induced spin structure on the bounding 3-manifold when the intersection pairing was even.

The point of view in [B2] was to fix a closed, connected, oriented 3-manifold M and a symmetric, bilinear, integral pairing $\mathscr{L}$ on a free abelian group E, and then to consider the oriented homeomorphism classes of 1-connected 4-manifolds with boundary M and intersection pairing isomorphic to $\mathscr{L}$. Invariants were constructed which distinguished these classes and it was also shown that in many instances all the potential values of these invariants were realized by appropriate manifolds. One goal of this paper is to show that it is always the case that these invariants assume their full range of values. In doing this we shall recast some of the work of [B2]. The main interest in these classification results is the tools they provide for studying 4-manifolds. In particular they have been used to study the representation of 2-dimensional homology classes by topologically locally-flat surfaces ([B1], [S1], [S2]). In the final two sections of this article we apply our constructions to prove some existence and uniqueness results on such representations. We now describe more fully the contents of this paper. Precise definitions may be found in §1.

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* Partially supported by NSERC grant A7819 and FCAR grant EQ3518.

Let M^3 be a 3-manifold as described above and $(E, \mathscr{L})$ a bilinear form space presenting $H_*(M)$ (Definition (1.3)). The group $A(M)$ of homological isometries of M (Definition (1.4)) acts transitively and effectively on the left of the set of all presentations of $H_*(M)$ by $(E, \mathscr{L})$. When $(E, \mathscr{L})$ is an even pairing, each presentation $P : (E, \mathscr{L}) \to H_*(M)$ determines a distinguished set of spin structures $\mathrm{Spin}_P(M) \subseteq \mathrm{Spin}(M)$, which is an orbit of the natural action of $I^1(M) = \text{image}\,(H^1(M) \to H^1(M; \mathbb{Z}/2))$ on $\mathrm{Spin}(M)$. A *marked presentation* P_* of $H_*(M)$ is a pair consisting of a presentation $P : (E, \mathscr{L}) \to H_*(M)$ and a *marking* of P, being

(i) an element of $\mathbb{Z}/2$ when $(E, \mathscr{L})$ is odd;
(ii) an element of $\mathrm{Spin}_P(M)$ when $(E, \mathscr{L})$ is even.

In the odd case, the group $A(M) \times \mathbb{Z}/2$ acts transitively and effectively on the left of the set of marked presentations. In the even case, there is an extension $\hat{A}(M)$ of $I^1(M)$ by $A(M)$ which so acts (Definition (1.8)).

A *topological realization* of a given marked presentation of $H_*(M)$ is a compact, simply-connected, oriented 4-manifold V with boundary M such that

(i) there is an isometry $\Lambda : (E, \mathscr{L}) \to (H_2(V), \cdot)$ such that with respect to the given presentations, $\partial(\Lambda) = 1_{H_*(M)}$ (see §1);
(ii) when $(E, \mathscr{L})$ is odd, the Kirby–Siebenmann invariant $ks(V)$ equals the marking of the presentation;
(iii) when $(E, \mathscr{L})$ is even, the unique spin structure on M extending over V equals the marking of the presentation.

THEOREM A. *Each marked presentation of $H_*(M)$ by a pairing $(E, \mathscr{L})$ is realized topologically. Any two realizations are homeomorphic by a homeomorphism extending the identity function on M.* □

Let $\mathscr{V}^0_{\mathscr{L}}(M)$ be the set of classes of compact, 1-connected, oriented 4-manifolds with boundary M and intersection pairing isomorphic to $(E, \mathscr{L})$, where two such manifolds are considered equivalent if they are homeomorphic by a homeomorphism restricting to 1_M. Theorem A defines a function from the set of marked presentations of $H_*(M)$ by $(E, \mathscr{L})$ to $V^0_{\mathscr{L}}(M)$. Now each 4-manifold V representing a class in $\mathscr{V}^0_{\mathscr{L}}(M)$ determines such a marked presentation, and thus the function is surjective. Further, two marked presentations have the same image if and only if they have the same markings and there is an isometry between the two presentations whose boundary is $1_{H_*(M)}$. As these conditions are preserved under the action of $A(M) \times \mathbb{Z}/2$ when $\mathscr{L}$ is odd, and by $\hat{A}(M)$ when $\mathscr{L}$ is even, these actions will descend to transitive actions on $\mathscr{V}^0_{\mathscr{L}}(M)$. Hence we may identify $\mathscr{V}^0_{\mathscr{L}}(M)$ with the left cosets of some subgroup of $A(M) \times \mathbb{Z}/2$ in the odd case, and with the left cosets of some subgroup of $\hat{A}(M)$ in the even case. Explicitly, fix a 4-manifold V_0 representing an element of $\mathscr{V}^0_{\mathscr{L}}(M)$ and let P_* be a marked presentation realized by V_0. In the odd case, the stabilizer of the class of V is $A_p(M)$, the homological isometries of $H_*(M)$ induced by isometries of P (Definition (1.6)). In the even case, the stabilizer of the class of V is $\hat{A}_{P_*(M)}$, consisting of those pairs $(\alpha, \pi) \in \hat{A}(M)$ such that $\alpha \in A_p(M)$ and π fixes the marking. The resulting bijection between $\mathscr{V}^0_{\mathscr{L}}(M)$ and the cosets may be described by the function

$$c^\circ_P : \mathscr{V}^0_{\mathscr{L}}(M) \to A(M)/A_P(M) \qquad \text{(Definition (1.6))}$$

and, when $\mathscr{L}$ is even, by the function

$$\hat{c}^\circ_{P_*}(M) : \mathscr{V}^0_{\mathscr{L}}(M) \to \hat{A}(M)/\hat{A}_{P_*(M)} \qquad \text{(after Lemma (1.10)).}$$

THEOREM B.
(a) *If* $(E, \mathscr{L})$ *is odd,* $c^\circ_P \times ks : \mathscr{V}^0_{\mathscr{L}}(M) \to A(M)/A_P(M) \times \mathbb{Z}/2$ *is a bijection;*
(b) *If* $(E, \mathscr{L})$ *is even,* $\hat{c}^\circ_{P_*} : \mathscr{V}^0_{\mathscr{L}}(M) \to \hat{A}(M)/\hat{A}_P(M)$ *is a bijection.* □

If M is a $\mathbb{Q}$-homology 3-sphere then $I^1(M) = 0$. In this case $\hat{A}(M) = A(M)$ and $\hat{A}_{P_*}(M) = A_P(M)$. We deduce:

COROLLARY C. *If M is a $\mathbb{Q}$-homology 3-sphere then*
(a) $c^\circ_P \times ks : \mathscr{V}^0_{\mathscr{L}}(M) \to A(M)/A_P(M) \times \mathbb{Z}/2$ *is a bijection when* $(E, \mathscr{L})$ *is odd;*
(b) $c^\circ_P : \mathscr{V}^0_{\mathscr{L}}(M) \to A(M)/A_P(M)$ *is a bijection when* $(E, \mathscr{L})$ *is even.* □

We remark that $A(M)/A_P(M) \cong A^t(M)/A^t_P(M)$ where $A^t(M)$ is the group of link-pairing preserving isomorphisms of $T_1(M)$ and $A^t_P(M)$ is the subgroup of $A^t(M)$ obtained by restricting the elements of $A_P(M)$ to $T_1(M)$ (see Theorem (1.7)). Similarly $\hat{A}(M)/\hat{A}_{P_*}(M)$ is determined as a quotient $\hat{A}^t(M)/\hat{A}^t_{P_*}(M)$ of a finite group $\hat{A}^t(M)$.

If f is an orientation preserving homeomorphism of M and V represents a class in $\mathscr{V}^0_{\mathscr{L}}(M)$, then the 4-manifold $V_f = M \times I \cup_{(x,0)=f(x)} V$ does also. This determines a left action of $\mathscr{H}_+(M)$, the group of orientation preserving homeomorphisms of M, on $\mathscr{V}^0_{\mathscr{L}}(M)$. Now V' is in the class of V_f if and only if there is a homeomorphism $F : V' \to V$ restricting to f on M. Thus, the orbits of this action correspond to $\mathscr{V}_{\mathscr{L}}(M)$, which denotes the orientation preserving homeomorphism classes of compact, 1-connected, oriented 4-manifolds with boundary M and intersection pairing isomorphic to $(E, \mathscr{L})$. Now under the identifications of Theorem B, the action of f on $\mathscr{V}^0_{\mathscr{L}}(M)$ corresponds to the action of $f_*^{-1} \times 1_{\mathbb{Z}/2}$ on $A(M)/A_P(M) \times \mathbb{Z}/2$ in the odd case, and to $f_*^{-1} \times f_\#^{-1}$ on $\hat{A}(M)/\hat{A}_{P_*}(M)$ in the even case (here, $f_\#^{-1}$: Spin $(M) \to$ Spin (M) is the natural function induced by f^{-1}). Thus if we set

(i) $H_+(M) = \{f_* \mid f \in \mathscr{H}_+(M)\}$ and $B_P(M) = H_+(M)\backslash A(M)/A_P(M)$;
(ii) $\hat{H}_+(M) = \{(f_*, f_\#) \mid f \in \mathscr{H}_+(M)\}$ and $\hat{B}_{P_*}(M) = \hat{H}_+(M)\backslash \hat{A}(M)/\hat{A}_{P_*}(M)$,

and let c_P and $\hat{c}_{P_*}$ denote the reductions of c_P° and $\hat{c}_{P_*}^\circ$ we obtain

THEOREM D.
(a) *If $(E, \mathscr{L})$ is odd, $c_P \times ks : \mathscr{V}_{\mathscr{L}}(M) \to B_P(M) \times \mathbb{Z}/2$ is a bijection;*
(b) *If $(E, \mathscr{L})$ is even, $\hat{c}_{P_*} : \mathscr{V}_{\mathscr{L}}(M) \to \hat{B}_{P_*}(M)$ is a bijection.* □

COROLLARY E. *If M is a $\mathbb{Q}$-homology 3-sphere*
(a) *$\mathscr{V}_{\mathscr{L}}(M) \cong B_P(M) \times \mathbb{Z}/2$ when $(E, \mathscr{L})$ is odd;*
(b) *$\mathscr{V}_{\mathscr{L}}(M) \cong B_P(M)$ when $(E, \mathscr{L})$ is even.* □

In the final two sections of the paper we apply the work above to prove certain existence and uniqueness theorems for representations of primitive homology classes in 1-connected, oriented 4-manifolds by closed surfaces. More precisely, let W be such a 4-manifold and $\xi_0 \in H_2(M)$ a primitive homology class. Define $\Theta(\xi_0) \in \mathbb{Z}/2$ to be the quantity

$$\Theta(\xi_0) \equiv \begin{cases} ks(W) + \frac{1}{8}[\text{signature }(W) - \xi_0 \cdot \xi_0] & \text{if } \xi_0 \text{ is characteristic,} \\ 0 & \text{otherwise.} \end{cases}$$

We say that an oriented surface $F \subseteq W$ gives a *simple representation* of ξ_0 if F is locally-flat in W with 1-connected complement and the fundamental class of F represents ξ_0.

THEOREM F. *If $\Theta(\xi_0) \equiv 0$, then for each $g \geq 0$ there is a surface of genus g in W giving a simple representation of ξ_0. Further, such a surface is unique up to ambient isotopy in W.*
If $\Theta(\xi_0) \equiv 1$, then the same result holds for each $g \geq 1$. □

When $\Theta(\xi_0) \equiv 0$, the existence part of Theorem F is due to Lee and Wilczynski. They also prove some uniqueness results when $g = 0$. Their method is to apply 4-dimensional surgery theory and is completely different from those we use here. See [LW] for more details.

The necessity of the condition $\Theta(\xi_0) \equiv 0 \pmod 2$ for the realization of ξ_0 by a locally flat 2-sphere originates in the work of Kervaire and Milnor [KM]. For nonprimitive classes, other necessary conditions arise out of the G-signature theorem (see [Tr], [R] and [HS]). Lee and Wilczynski [LW] have shown that in quite general circumstances, these conditions are also sufficient to represent a homology class by a locally flat 2-sphere.

The uniqueness statements in Theorem F may be elaborated on. Let F be a surface of genus $g \geq 0$ and let $\mathcal{I}_g$ denote the collection of ambient isotopy classes of embeddings in W of F which give simple representations of ξ_0. Next set $r = \xi_0 \cdot \xi_0$ and let $D(F, r)$ be the 2-disc bundle of euler class r over F. Let $\tilde{\mathcal{I}}_g$ denote the collection of flat embeddings of $D(F, r)$ in W which, when restricted to F, give simple representations of ξ_0. Finally let $E = \{\xi \in H_2(W) \mid \xi \cdot \xi_0 = 0\}$ and denote by $\mathcal{L}$ the restriction of the intersection pairing on $H_2(W)$ to E.

THEOREM G.
(a) *If $\mathcal{L}$ is odd, both $\mathcal{I}_g$ and $\tilde{\mathcal{I}}_g$ are singletons;*
(b) *If $\mathcal{L}$ is even but ξ_0 is not characteristic (i.e. W is spin), then $\mathcal{I}_g$ is a singleton but $\tilde{\mathcal{I}}_g$ has 2^{2g} elements;*
(c) *If $\mathcal{L}$ is even and ξ_0 is characteristic, then both $\mathcal{I}_g$ and $\tilde{\mathcal{I}}_g$ have $[2^g + (-1)^{\Theta(\xi_0)}2^{g-1}]$ elements, where $[\cdot]$ is the greatest integer less than or equal to function.* $\square$

The paper is organized as follows. Section 1 contains definitions and conventions. Section 2 contains the proof of Theorem A. Section 3 contains the proof of the existence part of Theorem F and finally in Section 4 we complete the proof of Theorem F and prove Theorem G.

The author gratefully acknowledges useful conversations with Ian Hambleton concerning the material in §3 and §4.

§1. Notations and Terminology

We shall assume throughout this article that all manifolds are compact and oriented. Further all homeomorphisms will be assumed to preserve orientations. Boundaries of manifolds will have the orientation corresponding to the boundary of the fundamental class of the manifold they bound. As in the introduction, M will denote a closed, connected 3-manifold and $T_1(M)$ the torsion subgroup of $H_1(M)$. The free part of $H_1(M)$ is the quotient $F_1(M) = H_1(M)/T_1(M)$.

If C_1 and C_2 are disjoint 1-cycles in M representing classes $[C_1]$ and $[C_2]$ in $T_1(M)$, there is a rational value linking number $\ell_{\mathbb{Q}}(C_1, C_2) \in \mathbb{Q}$ ([ST], §77). The torsion pairing

$$\ell_{\mathbb{Q}/\mathbb{Z}} : T_1(M) \times T_1(M) \to \mathbb{Q}/\mathbb{Z}$$

is just the (mod $\mathbb{Z}$) reduction $\ell_{\mathbb{Q}}$:

$$\ell_{\mathbb{Q}/\mathbb{Z}}([C_1], [C_2]) \equiv \ell_{\mathbb{Q}}(C_1, C_2) \pmod{\mathbb{Z}}.$$

We shall assume that all links L in M are tame. $T(L)$ will denote a closed tubular neighbourhood of L. Given a knot K in M, let μ be its meridian. More precisely, let D be a small 2-disc fibre of $T(K)$ oriented so that it intersects K positively. Then μ will be the boundary of D.

When M is a $\mathbb{Q}$-homology 3-sphere there is a canonical isomorphism $\ell_{\mathbb{Q}}(K, \ ) : H_1(M\backslash K; \mathbb{Q}) \to \mathbb{Q}$ determined by the requirement that $[\mu] \mapsto 1$.

(1.1) DEFINITION. The *longitude* of a knot K in a $\mathbb{Q}$-homology 3-sphere M is the unique class $\lambda \in H_1(\partial T(K); \mathbb{Q})$ satisfying (i) $\mu \cdot \lambda = 1$ in $H_1(\partial T(K); \mathbb{Q})$, and (ii) the image of λ in $H_1(M\backslash K; \mathbb{Q})$ is zero.

Note that the first condition is equivalent to λ being rationally homologous to K in $T(K)$ while the second is equivalent to $\ell_{\mathbb{Q}}(K, \lambda) = 0$.

It is shown in [BL2] that when M is a $\mathbb{Q}$-homology 3-sphere, a class $p\mu + q\lambda \in H_1(\partial T(K); \mathbb{Q})$ is represented by an essential simple closed curve on $\partial T(K)$ if and only if $q \in \mathbb{Z}$ and there is a $c \in \mathbb{Z}$ coprime with q such that $p \equiv c - q\ell_{\mathbb{Q}/\mathbb{Z}}(K, K) \pmod{\mathbb{Z}}$. Oriented Dehn surgeries along K are classified by such pairs (p, q) (see §1 of [BL2]).

(1.2) DEFINITION. An *integral framing* of a knot K in a 3-manifold M is one for which the framing curve is isotopic to $\pm K$ in $T(K)$. An *integrally framed link* $\mathbb{L}$ in M is a framed link for which each framing curve is integral.

Denote by $\chi(\mathbb{L})$ the manifold obtained by performing the surgery prescribed by $\mathbb{L}$. $\chi(\mathbb{L})$ has the orientation extending that of $M\backslash \mathring{T}(L)$.

Note that when M is a $\mathbb{Q}$-homology 3-sphere, we may express an integrally framed link $\mathbb{L}$ as $\mathbb{L} = K_1^{(p_1, \varepsilon_1)} \cup \cdots \cup K_n^{(p_n, \varepsilon_n)}$ where $\varepsilon_i \in \{\pm 1\}$ and $p_i \equiv \varepsilon_i \ell_{\mathbb{Q}/\mathbb{Z}}(K_i, K_i)$ (mod $\mathbb{Z}$) for each $i \in \{1, 2, \ldots, n\}$. In this case set

$$\ell_{ij} = \ell_{\mathbb{Q}}(K_i, K_j) \qquad i \neq j,$$

and define the framing matrix of $\mathbb{L}$ as

$$B_{\mathbb{L}} = \begin{bmatrix} p_1 & & & \\ & p_2 & & \varepsilon_j \ell_{ij} \\ & & \ddots & \\ & \varepsilon_j \ell_{ij} & & \\ & & & p_n \end{bmatrix}$$

It is shown in [BL2] that

$$|B_{\mathbb{L}}| = \pm |H_1(\chi(\mathbb{L}))| / |H_1(M)|.$$

A *bilinear form space* is a pair $(E, \mathscr{L})$ where E is a finitely generated free abelian group and $\mathscr{L} : E \times E \to \mathbb{Z}$ is a symmetric bilinear pairing. For instance if V is a 1-connected 4-manifold, the intersection pairing $(H_2(V), \cdot)$ is such a pairing.

A form $(E, \mathscr{L})$ is called *even* if $\mathscr{L}(\xi, \xi) \equiv 0 \pmod 2$ for each $\xi \in E$ and is called *odd* otherwise.

An *isometry* of $(E, \mathscr{L})$ is an isomorphism of E which preserves $\mathscr{L}$.

Denote by E^* the $\mathbb{Z}$-dual of $E : E^* = \operatorname{Hom}(E, \mathbb{Z})$.

(1.3) DEFINITION. A *presentation* of $H_*(M)$ by a bilinear form space $(E, \mathscr{L})$ is an exact sequence

$$0 \longrightarrow H_2(M) \xrightarrow{h} E \xrightarrow{ad(\mathscr{L})} E^* \xrightarrow{\partial} H_1(M) \longrightarrow 0$$

such that

(i) if $ad(\mathscr{L})(\xi_i) = m_i \eta_i \ (i = 1, 2)$ where $m_1 m_2 \neq 0$, then $\ell_{\mathbb{Q}/\mathbb{Z}}(\partial\eta_1, \partial\eta_2) \equiv -(m_1 m_2)^{-1} \mathscr{L}(\xi_1, \xi_2) \pmod{\mathbb{Z}}$;

(ii) if $\beta \in H_2(M)$ and $\eta \in E^*$ then $(\partial\eta) \cdot \beta = \eta(h(\beta))$.

For instance if M is the boundary of a 4-manifold V then it is well known that $(H_2(V), \cdot)$ determines a presentation P_V of $H_*(M)$ (see §3 of [GL]).

(1.4) DEFINITION. A *homological isometry* between two closed 3-manifolds M and M' is a pair of isomorphisms $\alpha = (\alpha_1, \alpha_2)$ where $\alpha_i : H_i(M) \to H_i(M')$ $(i = 1, 2)$ which satisfy

(i) if $\upsilon_1, \upsilon_2 \in T_1(M)$ then $\ell'_{\mathbb{Q}/\mathbb{Z}}(\alpha_1(\upsilon_1), \alpha_1(\upsilon_2)) \equiv \ell_{\mathbb{Q}/\mathbb{Z}}(\upsilon_1, \upsilon_2)$;

(ii) if $\upsilon \in H_1(M)$ and $\beta \in H_2(M)$ then $\alpha_1(\upsilon) \cdot \alpha_2(\beta) = \upsilon \cdot \beta$.

When $M = M'$ we call α an *automorphism*. $A(M)$ will denote the group of all such automorphisms.

Let P be a presentation of $H_*(M)$, as in Definition (1.3), and let $\alpha \in A(M)$. We can define a new presentation of $H_*(M)$, $\alpha(P)$, by replacing ∂ by $\alpha_1 \circ \partial$ and h by $h \circ \alpha_2^{-1}$. It is an easy exercise to verify that this defines a transitive, effective, left action of $A(M)$ on the set of presentations of $H_*(M)$ by a bilinear form space $(E, \mathscr{L})$.

(1.5) LEMMA. *If* $(E, \mathscr{L})$ *is an odd form presenting* $H_*(M)$, *then the action of* $A(M) \times \mathbb{Z}/2$ *on the set of marked presentations of* $H_*(M)$ *given by*

$$(\alpha, m) \cdot (P, n) = (\alpha(P), m + n),$$

is both transitive and effective. $\square$

Any orientation preserving homeomorphism f of M, induces an element $f_* \in A(M)$ and we let $H_+(M)$ be the subgroup of $A(M)$ consisting of all such automorphisms.

An *isometry* between presentations $(E, \mathscr{L})$ of $H_*(M)$ and $(E', \mathscr{L}')$ of $H_*(M')$ is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_2(M) \arrow[r, "h"] \arrow[d, "\alpha_2"'] & E \arrow[r, "ad(\mathscr{L})"] \arrow[d, "\Lambda"'] & E^* \arrow[r, "\partial"] & H_1(M) \arrow[r] \arrow[d, "\alpha_1"] & 0 \\
0 \arrow[r] & H_2(M') \arrow[r, "h'"] & E' \arrow[r, "ad(\mathscr{L}')"] & (E')^* \arrow[r, "\partial'"] \arrow[u, "\Lambda^*"'] & H_1(M') \arrow[r] & 0
\end{tikzcd}$$

where Λ is an isometry of the pairings. Note that Λ determines completely the isometry and so in particular we shall write (somewhat ambiguously) $(\alpha_1, \alpha_2) = \partial(\Lambda)$. As Λ is an isometry it follows that $\partial(\Lambda)$ is a homological isometry between M and M'.

(1.6) DEFINITION. Given a presentation $P : (E, \mathscr{L}) \to H_*(M)$, let $A_P(M)$ be the subgroup of $A(M)$ defined by

$$A_P(M) = \{\partial(\Lambda) |\ \Lambda \text{ is an isometry of } P\}$$

The function $c^0_{\mathscr{L}} : \mathscr{V}^0_{\mathscr{L}}(M) \to A(M)/A_P(M)$ given by $c^0_{\mathscr{L}} = \partial(\Lambda)A_P(M)$, where $\Lambda : P \to P_V$ is an isometry, is easily seen to be well-defined.

Define

$$B_P(M) = H_+(M) \backslash A(M)/A_P(M).$$

It was shown in [B2] that the quotients $A(M)/A_P(M)$ and $B_P(M)$ depend only on torsion information. To describe this let

(i) $A'(M)$ be the set of $\ell_{\mathbb{Q}/\mathbb{Z}}$-preserving isomorphisms of $T_1(M)$;
(ii) $A'_P(M)$ be the subgroup of $A'(M)$ consisting of those $\ell_{\mathbb{Q}/\mathbb{Z}}$-preserving isomorphisms induced from isometries of $(E, \mathscr{L})$;
(iii) $H'_+(M) = \{f_* : T_1(M) \to T_1(M) | f \in \mathscr{H}_+(M)\}$;
(iv) $B'_P(M) = H'_+(M)\backslash A'(M)/A'_P(M)$.

Note that $F_1(M)$ and $H_1(M)$ are dually paired by the intersection pairing and thus the restriction homomorphism $A(M) \to A'(M)$ is surjective. Evidently this homomorphism takes $A_P(M)$ onto $A'_P(M)$ and $H_+(M)$ onto $H'_+(M)$. The following theorem is a combination of results from §1 of [B2]. For $\alpha \in A(M)$ let $\alpha' \in A'(M)$ be its restriction.

(1.7) THEOREM. *An automorphism α lies in $A_P(M)$ if and only if α' lies in $A'_P(M)$. As a consequence, the restriction induces bijections*

$$A(M)/A_P(M) \to A'(M)/A'_P(M) \quad \textit{and} \quad B_P(M) \to B'_P(M). \qquad \square$$

To deal with the case where $(E, \mathscr{L})$ is even, we need to enrich $A(M)$. For a spin structure σ on M, let $q_\sigma : T_1(M) \to \mathbb{Q}/\mathbb{Z}$ be the associated quadratic enhancement of $\ell_{\mathbb{Q}/\mathbb{Z}}$ (see [T] for instance). If P is a presentation of $H_*(M)$, as in Definition (1.3), by an even pairing $(E, \mathscr{L})$, then P determines a quadratic enhancement q_P of $\ell_{\mathbb{Q}/\mathbb{Z}}$: for $\xi \in E$ such that $ad(\mathscr{L})(\xi) = m\eta$ where $m \neq 0$, then the reduction of $(-1/2m^2)\mathscr{L}(\xi, \xi) \pmod{\mathbb{Z}}$ defines $q_P(\partial\eta)$. The set $\mathrm{Spin}_P(M)$ of spin structures σ on M with $q_\sigma = q_P$ is precisely an $I^1(M)$-orbit where $I^1(M) =$ image $(H^1(M) \to H^1(M; \mathbb{Z}/2)) \subseteq H^1(M; \mathbb{Z}/2)$. If M is the boundary of a compact, 1-connected, spin 4-manifold V, and P_V is the presentation of $H_*(M)$ from the homology sequence of (V, M), the spin structure on V restricts to an element σ_V of $\mathrm{Spin}_P(M)$.

Let $\alpha \in A(M)$ and suppose that π is a permutation of $\mathrm{Spin}(M)$. We shall say that π is *α-adapted* if

(i) $\pi(\alpha_1^*(x) \cdot \sigma) = x \cdot \pi(\sigma)$ for each $x \in H^1(M; \mathbb{Z}/2)$ and $\sigma \in \mathrm{Spin}(M)$;
(ii) $q_{\pi(\sigma)}(\alpha_1(z)) = q_\sigma(z)$ for each $z \in T_1(M)$ and $\sigma \in \mathrm{Spin}(M)$.

(1.8) DEFINITION. Let $\hat{A}(M)$ be the set of all pairs (α, π) where $\alpha \in A(M)$ and π is α-adapted.

Note that $\hat{A}(M)$ is a group under coordinate-wise composition. It is shown in [B2] (Lemma (3.3)) that there is an exact sequence

$$1 \longrightarrow I^1(M) \longrightarrow \hat{A}(M) \xrightarrow{p} A(M) \longrightarrow 1 \tag{1.9}$$

where p is the natural projection. $I^1(M)$ includes in $\hat{A}(M)$ owing to the fact that a permutation adapted to the identity is just the translation in Spin (M) by an element of $I^1(M)$ (see §3 of [B2]).

It is relatively straightforward to verify that the formula $(\alpha, \pi) \cdot (P, \sigma) = (\alpha(P), \pi(\sigma))$ defines a left action of $\hat{A}(M)$ on the set of marked presentations of $H_*(M)$ by an even form $(E, \mathscr{L})$. Indeed we have the following,

(1.10) LEMMA. *The action $(\alpha, \pi) \cdot (P, \sigma) = (\alpha(P), \pi(\sigma))$ defines an effective, and transitive action of $\hat{A}(M)$ on the set of marked presentations of $H_*(M)$ by an even form $(E, \mathscr{L})$.* □

Suppose that P_* is a marked presentation of $H_*(M)$ by an even form $(E, \mathscr{L})$, say $\sigma \in \mathrm{Spin}_P(M)$ is the marking. For each isometry Λ of P, let π_Λ be the $\partial(\Lambda)$-adapted permutation of Spin (M) determined by $\pi_\Lambda(\sigma) = \sigma$. There is a subgroup of $\hat{A}(M)$

$$\hat{A}_{P_*}(M) = \{(\partial(\Lambda), \pi_\Lambda) \mid \Lambda \text{ is an isometry of } P\}.$$

The function $\hat{c}^\circ_{P_*} : \mathscr{V}^\circ_{\mathscr{L}}(M) \to \hat{A}(M)/\hat{A}_{P_*}(M)$ given by $\hat{c}^\circ_{P_*}(V) = (\partial(\Lambda), \pi_\Lambda)$, where $\Lambda : P \to P$ is an isometry, is easily seen to be well-defined. Further, each $f \in H_+(M)$ determines an f_*-adapted permutation $f_\#$ of Spin (M). Thus there is another natural subgroup of $\hat{A}(M)$

$$\hat{H}_+(M) = \{(f_*, f_\#) \mid f \in H_+(M)\}.$$

Define

$$\hat{B}_{P_*}(M) = \hat{H}_+(M) \backslash \hat{A}(M)/\hat{A}_{P_*}(M).$$

We remark that as in Theorem (1.6), the quotients $\hat{A}(M)/\hat{A}_{P_*}(M)$ and $\hat{B}_{P_*}(M)$ are determined by torsion information. The reader is directed to [B2] for the details.

§2. Realization of simply-connected 4-manifolds

In this section we shall prove Theorem A. We shall divide the proof of existence into two cases according to whether $H_1(M)$ is finite or not.

Case 1 ($H_1(M; \mathbb{Q}) \cong 0$). Let $P : (E, \mathscr{L}) \to H_*(M)$ be a presentation, say

$$0 \longrightarrow E \xrightarrow{ad(\mathscr{L})} E^* \xrightarrow{\partial} H_1(M) \longrightarrow 0.$$

As $ad(\mathscr{L})(E)$ has finite index in E^*, the pairing $(ad(\mathscr{L})(\xi_1), ad(\mathscr{L})(\xi_2)) = \mathscr{L}(\xi_1, \xi_2)$ extends uniquely to a symmetric rational valued pairing $\mathscr{L}^*$ on E^*. Explicitly, if $\eta_1, \eta_2 \in E^*$ and $m_1, m_2 > 0$ are chosen so that $m_1\eta_1, m_2\eta_2 \in ad(\mathscr{L})(E)$, then we set

$$\mathscr{L}^*(\eta_1, \eta_2) = \frac{1}{m_1 m_2} \mathscr{L}(ad(\mathscr{L})^{-1}(m_1\eta_1), ad(\mathscr{L})^{-1}(m_2\eta_2)).$$

Note that the following identities hold:

$$\mathscr{L}^*(\eta, ad(\mathscr{L})(\xi)) = \eta(\xi) \qquad \forall \xi \in E \text{ and } \eta \in E^*. \tag{2.1}$$

$$\ell_{\mathbb{Q}/\mathbb{Z}}(\partial\eta_1, \partial\eta_2) \equiv -\mathscr{L}^*(\eta_1, \eta_2) \pmod{\mathbb{Z}} \qquad \forall \eta_1, \eta_2 \in E^*. \tag{2.2}$$

Fix a basis $\xi_1, \ldots, \xi_n$ for E and let $\eta_1, \ldots, \eta_n$ be the dual basis for E^*: $\eta_i(\xi_j) = \delta_{ij}$.

(2.3) LEMMA. *The matrix of $\mathscr{L}^*$ with respect to the basis $\eta_1, \ldots, \eta_n$ of E^* is the inverse of the matrix of $\mathscr{L}$ with respect to the basis $\xi_1, \ldots, \xi_n$.*

Proof. First note that the matrix of $ad(\mathscr{L})$ with respect to the given bases is $(\mathscr{L}(\xi_i, \xi_j))^{n \times n}$.

Extend $\mathscr{L}$ to $E \otimes \mathbb{Q}$ in the obvious way. Then we may consider $ad(\mathscr{L})$ as an isomorphism $ad(\mathscr{L}) : E \otimes \mathbb{Q} \to (E^*) \otimes \mathbb{Q}$. Note that identity (2.1) continues to hold. Let $(a_{ij})^{n \times n}$ be the $n \times n$ rational matrix which is inverse to the matrix of $\mathscr{L}$ with respect to the given basis. Then

$$\begin{aligned} \mathscr{L}^*(\eta_i, \eta_j) &= \eta_i(ad(\mathscr{L})^{-1}(\eta_j)) \qquad \text{by (2.1).} \\ &= \eta_i\left(\sum_{k=1}^{n} a_{kj}\xi_j\right) \\ &= a_{ij}. \end{aligned}$$

□

Let $L = K_1 \cup \cdots \cup K_n$ be an oriented link in M such that $[K_i]$, the class of K_i in $H_1(M)$, equals $\partial\eta_i$. According to identity (2.2) we have for $i \neq j$

$$\begin{aligned} -\mathscr{L}^*(\eta_i, \eta_j) &\equiv \ell_{\mathbb{Q}/\mathbb{Z}}([K_i], [K_j]) \pmod{\mathbb{Z}} \\ &\equiv \ell_{\mathbb{Q}}(K_i, K_j) \qquad \pmod{\mathbb{Z}}. \end{aligned}$$

Now passing a strand of K_i across a strand of K_j alters $\ell_{\mathbb{Q}}(K_i, K_j)$ by ± 1. Hence after a finite number of such isotopies we may suppose that L was chosen to satisfy

$$\ell_{\mathbb{Q}}(K_i, K_j) = -\mathscr{L}^*(\eta_i, \eta_j) \qquad i \neq j.$$

Next we determine preferred integral framings for the components of L. To do this we observe that

$$-\mathscr{L}^*(\eta_i, \eta_i) \equiv \ell_{\mathbb{Q}/\mathbb{Z}}([K_i], [K_j]) \pmod{\mathbb{Z}}$$

and thus the class

$$-\mathscr{L}^*(\eta_i, \eta_i)\mu_i + \lambda_i$$

represents a parallel to K_i (see §1). It therefore determines an integral framing of K_i, $1 \leq i \leq n$. Let $\mathbb{L}$ be the associated framed link. By construction, the framing matrix of $\mathbb{L}$ is given by

$$B_{\mathbb{L}} = -(\mathscr{L}^*(\eta_i, \eta_j))^{n \times n} = -(\mathscr{L}(\xi_i, \xi_j))^{-1}.$$

Hence

$$\frac{|H_1(\chi(\mathbb{L}))|}{|H_1(M)|} = \pm|B_{\mathbb{L}}| = \pm\frac{1}{|\mathscr{L}(\xi_i, \xi_j)|} = \frac{\pm 1}{|H_1(M)|}.$$

It follows that $\chi(\mathbb{L})$ is a $\mathbb{Z}$-homology 3-sphere. Define a compact 4-manifold V,

$$V = M \times I \cup [H_1^{(2)} \cup \cdots \cup H_n^{(2)}] \cup_{\chi(\mathbb{L})} W$$

where $H_i^{(2)}$ is a 2-handle attached to $M \times \{0\}$ according to the framing on K_i and W is the contractible 4-manifold with boundary $\chi(\mathbb{L})$ (see [F]). Turning the handles upside-down we see that V is just W with n 2-handles attached. Hence $V \simeq \bigvee_{i=1}^n S^2$ and so in particular $\pi_1(V) \cong 1$ and $H_2(V) \cong \mathbb{Z}^n$.

Next we describe a natural isometry $\Lambda : (E, \mathscr{L}) \to (H_2(V), \cdot)$. Consider the basis $\{\eta_1', \eta_2', \ldots, \eta_n'\}$ of $H_2(V, M)$ where η_i' is represented by a core of the ith 2-handle $H_i^{(2)}$ in V. Define an isomorphism $\Lambda^* : H_2(V, M) \to E^*$ by setting $\Lambda^*(\eta_i') = \eta_i$, $1 \leq i \leq n$. Finally let $\Lambda : E \to H_2(V)$ be the isomorphism dual to Λ^*.

(2.4) LEMMA. $\Lambda : (\mathbb{E}, \mathscr{L}) \to (H_2(V), \cdot)$ *is an isometry.*

Proof. We describe more precisely the isomorphism Λ.

Let $\mathbb{L}'$ be the framed link in $\chi(\mathbb{L})$ inverse to $\mathbb{L}$. The ith component of $\mathbb{L}'$, K_i' say, is the boundary of the cocore of $H_i^{(2)}$ and is oriented so that its meridian represents the class $-\mathscr{L}^*(\eta_i, \eta_i)\mu_i + \lambda_i \in H_1(\partial T(K_i); \mathbb{Q}) = H_1(\partial T(K_i'); \mathbb{Q})$. Denote by $\{\xi_i', \xi_2', \ldots, \xi_n'\}$ the basis of $H_2(V)$ where ξ_i' is represented by a cycle consisting of a Seifert surface in $\chi(\mathbb{L})$ for K_i' and a cocore of $H_i^{(2)}$. Evidently this basis is dual to $\{\eta_1', \eta_2', \ldots, \eta_n'\}$ and hence $\Lambda(\xi_i) = \xi_i'$, $1 \le i \le n$. Now it is well known that for $i \ne j$, $\xi_i' \cdot \xi_j' = \ell_{\mathbb{Q}}(K_i', K_j')$. On the other hand, Lemma (1.5) of [BL2] shows $\ell_{\mathbb{Q}}(K_i', K_j') = -(B_{\mathbb{L}}^{-1})_{ij}$, and thus by Lemma (2.3), $\xi_i' \cdot \xi_j' = \mathscr{L}(\xi_i, \xi_j)$ for $i \ne j$. Finally $\xi_i' \cdot \xi_i'$ is determined by the framing on K_i', and one may use the results of §1 of [BL2] to deduce $\xi_i' \cdot \xi_i' = \mathscr{L}(\xi_i, \xi_i)$ for $1 \le i \le n$. Hence Λ is an isometry. $\square$

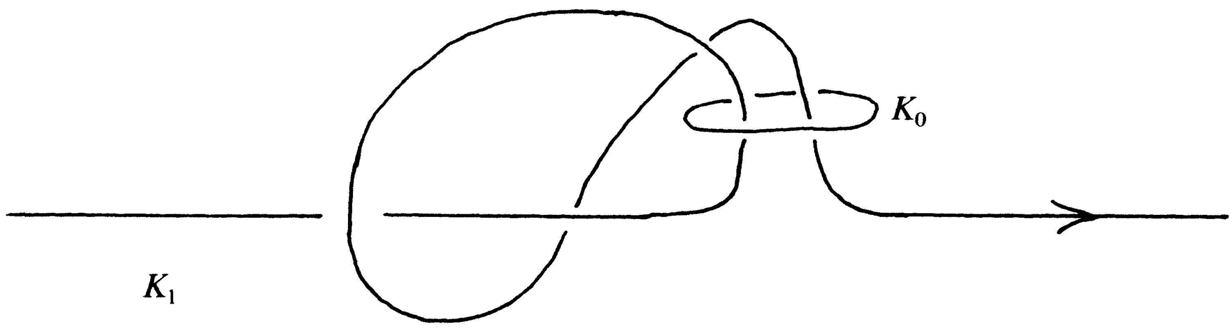
Now it is clear from the construction that with respect to the presentations P and P_V, $\partial(\Lambda) = 1_{H_*(M)}$. Thus to complete the existence part of Theorem A, it suffices to show how to vary the ks-invariant of V when $\mathscr{L}$ is odd and how to realize an arbitrary $\sigma \in \mathrm{Spin}_P(M)$ when $\mathscr{L}$ is even. But in the latter case, the hypothesis $H_1(M; \mathbb{Q}) = 0$ implies $I^1(M) = 0$. Hence there is a unique marking in $\mathrm{Spin}_P(M)$ and this must equal the restriction of the spin structure on V. Thus this case holds.

Suppose now that $\mathscr{L}$ is odd. Refer back to the definition of V. It is clear that $ks(V) \equiv ks(W) \equiv 1/8\mu(\chi(\mathbb{L}))$ (mod 2). It will therefore suffice to show how to alter $\mathbb{L}$ giving $\mathbb{L}'$ say, so that $\mu(\chi(\mathbb{L}')) \not\equiv \mu(\chi(\mathbb{L}))$. To that end we use the assumption that $\mathscr{L}$ is odd to choose an index $i \in \{1, 2, \ldots, n\}$ for which $\mathscr{L}(\xi_i, \xi_i) \equiv 1$ (mod 2). To simplify the notation below, we assume that $i = 1$. Consider the three auxiliary framed links described as follows,

(1) $\mathbb{L}_0 = K_2^{(p_2, 1)} \cup K_3^{(p_3, 1)} \cup \cdots \cup K_n^{(p_n, 1)}$ where $p_i = -\mathscr{L}^*(\eta_i, \eta_i)$.
(2) $\mathbb{L}' = K_1'^{(p_1, 1)} \cup \mathbb{L}_0$ where K_1' differs from K_1 only through the addition of a small trefoil:

K_1 K_1'

(3) $\mathbb{L}'' = K_0^{(1,1)} \cup \mathbb{L}$ where K_0 is positioned as indicated below:



Now by blowing down $K_0^{(1,1)}$ it is clear $\chi(\mathbb{L}') = \chi(\mathbb{L}'') = \chi(K_0^{(1,1)} \cup \mathbb{L}) = \chi(C^{(1,1)})$ where C is the knot in $\chi(\mathbb{L})$ corresponding to the image of K_0. Hence

$$\mu(\chi(\mathbb{L}')) \equiv \mu(\chi(C^{(1,1)})) \equiv \mu(\chi(\mathbb{L})) + 8arf(C) \pmod{16}.$$

The proof will be finished when we show $arf(C) \equiv 1 \pmod 2$. Now $arf(C)$ is congruent (mod 2) to the coefficient of z^2 in the Conway polynomial of C (see [Ka]). This latter quantity was calculated in Lemma (1.4) of [BL1] in the case where M is a $\mathbb{Z}$-homology 3-sphere. Switching to rational homology in that calculation, we see that the same argument works for M a $\mathbb{Q}$-homology 3-sphere. The result is that

$$arf(C) \equiv |B_{\mathbb{L}}^*|/|B_{\mathbb{L}}| \pmod 2$$

where

$$B_{\mathbb{L}}^* = \begin{bmatrix} 1 & \ell_{12} & \ell_{13} & \cdots & \ell_{1n} \\ 0 & & & & \\ 0 & & B_{\mathbb{L}_0} & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}.$$

Hence $|B_{\mathbb{L}}^*| = |B_{\mathbb{L}_0}| = \pm|H_1(\chi(\mathbb{L}_0))|/|H_1(M)|$. We have already noted that $|B_{\mathbb{L}}| = \pm 1/|H_1(M)|$ and thus $arf(C) \equiv |H_1(\chi(\mathbb{L}_0))| \pmod 2$. In the proof of Lemma (2.4), we proved that the framing curve for K_1' was the $(\mathscr{L}(\xi_1, \xi_1), 1)$ curve. Thus

$$\chi(\mathbb{L}_0) = \chi((K_1')^{(\mathscr{L}(\xi_1, \xi_1), 1)}).$$

Then $arf(C) \equiv L(\xi_1, \xi_1) \equiv 1 \pmod 2$ and this part of the proof of Theorem A is complete.

Case 2 (M arbitrary). Let $L = K_1 \cup K_2 \cup \cdots \cup K_m$ be an oriented link in M for which the classes $[K_1], [K_2], \ldots, [K_m]$ form a basis of $F_1(M) = H_1(M)/T_1(M)$. As $F_1(M)$ is dual to $H_2(M)$ via the intersection pairing, there are classes $\beta_1, \beta_2, \ldots, \beta_m \in H_2(M)$ satisfying $[K_i] \cdot \beta_j = \delta_{ij}$. It is well-known that each β_j may be represented by a flat surface S_j in M transverse to L. Hence, after tubing away pairs of oppositely signed intersection points between K_i and S_j, we may assume that K_i is transverse to S_j and further that

$$K_i \cap S_j = \begin{cases} \varnothing & i \neq j \\ x_i & i = j \end{cases}.$$

for $1 \leq i, j \leq m$. Removing a small disc neighbourhood of x_i in S_i shows that the meridian of K_i, μ_i say, is null-homologous in M_0, the exterior of L.

(2.5) LEMMA. *An integral surgery along L produces a* $\mathbb{Q}$*-homology* 3*-sphere* M_1 *as long as none of the surgeries is trivial, that is, none of the framing curves are meridians. Further* $H_1(M_1)$ *is naturally identified with* $T_1(M)$.

Proof. Consider the homology sequence of the pair (M, M_0),

$$\cdots \longrightarrow H_2(M, M_0) \xrightarrow{\partial} H_1(M_0) \longrightarrow H_1(M) \longrightarrow H_1(M, M_0) \longrightarrow \cdots.$$

The excision theorem implies that $H_1(M, M_0) = 0$ and that the image of ∂ is generated by the meridians of L. We have shown these curves to be null in $H_1(M_0)$ and therefore the homomorphism $H_1(M_0) \to H_1(M)$ is an isomorphism. This means that any collection of parallels to the components of L generate $F_1(M_0)$ while $T_1(M_0) = T_1(M)$. It follows that any non trivial integral surgery of M along L results in a 3-manifold M_1 with $H_1(M_1) = T_1(M_0) = T_1(M)$. □

Let $K(\mathscr{L}) = \ker(ad(\mathscr{L}))$. The presentation P determines a naturally defined non singular bilinear form space $(E/K(\mathscr{L}), \mathscr{L}')$ and presentation $P' : (E/K(\mathscr{L}), \mathscr{L}) \to H_*(M_1)$ (see §1 of [B2]). $(E, \mathscr{L})$ splits as an orthogonal sum $(E/K(\mathscr{L}), \mathscr{L}') \oplus (K(\mathscr{L}), 0)$. Denote by T the trace of an integral surgery as described in Lemma (2.5). Form a 4-manifold $V = T \cup_{M_1} V_1$ where V_1 is one of the spaces constructed in Case (1) which realizes P' and has boundary M_1. The calculations in the proof of Lemma (2.5) show that the inclusion $M_1 \to T$ induces an isomorphism $H_1(M_1) \to H_1(T)$. Hence, referring to the Mayer–Vietoris sequence stemming from the union $V = T \cup_{M_1} V_1$, we may conclude that $H_2(V) \cong H_2(T) \oplus H_2(V_1)$. The intersection pairing on V also follows this splitting and thus there is a canonical isomorphism

$$(H_2(V), \cdot) \cong (H_2(T), \cdot) \oplus (E/K(\mathscr{L}), \mathscr{L}').$$

Now examination of the constructions of T and V_1 reveals that V may be expressed as the union of $M \times I$ with n 2-handles and a contractible manifold. Hence $H_2(V) \cong \mathbb{Z}^n$. On the other hand, $H_2(M) \cong F_1(M)$ and so the intersection form an $H_2(V)$ has a summand of the form $(\mathbb{Z}^m, 0)$. As $(E/K(\mathscr{L}), \mathscr{L}')$ is non-singular, we conclude $(H_2(T), \cdot) \cong (\mathbb{Z}^m, 0)$. Clearly then $(H_2(V), \cdot) \cong (E, \mathscr{L})$. Further, we can produce an isometry $\Lambda : E \to H_2(V)$ with $\partial(\Lambda) = 1_{H_*(M)}$, just as in Case 1.

Suppose now that $\mathscr{L}$ is odd. By varying the ks-invariant of V_1, we vary the ks-invariant of V. This proves the existence part of Theorem A when $\mathscr{L}$ is odd.

Assume now that $\mathscr{L}$ is even. We must show how to vary the spin structure on V over the elements of $\mathrm{Spin}_P\,(M)$. To that end fix a nontrivial integral framing $\mathscr{F}_0$ on L as described in Lemma (2.5). Let M_1 be the associated $\mathbb{Q}$-homology 3-sphere and let T_0 denote the trace of this surgery. Let V_1 be chosen as before. For each $b \in \{0, 1\}^m$, let $\mathscr{F}_b$ be a nontrivial integral framing of L which agrees with $\mathscr{F}_0$ on K_i when $b_i = 0$ and differs from it by a meridinal twist otherwise. For each b let T_b denote the trace of the surgery prescribed by $\mathscr{F}_b$. Define $V_b = T_b \cup_{M_1} V_1$, $b \in \{0, 1\}^m$, and note that if σ_b denotes the unique element of Spin (M) extending over V_b, then these spin structures are distinct. Indeed if $b' \neq b_i$, then $\mathscr{F}_b$ and $\mathscr{F}_b$ differ by meridinal twist along K_i and thus σ_b, which extends over the ith 2-handle of T_b, does not extend over the ith 2-handle $T_{b'}$. Alternatively, we may argue that as the components of L represent a basis for $F_1(M)$, then $\{\sigma_b \mid b \in \{0, 1\}^m\}$ is the orbit $I^1(M) \cdot \sigma_0$. Clearly then, each marking of the presentation P is realized. This completes the proof of the existence part of Theorem A when $\mathscr{L}$ is even.

Proof of uniqueness in Theorem A. Given two manifolds V_1 and V_2 realizing a marked presentation of $H_*(M)$, there is a morphism $(1_M, \Lambda) : V_1 \to V_2$ (see [B2]). If $\mathscr{L}$ is odd, this may be replaced by a morphism $(1_M, \Lambda') : V_1 \to V_2$ with $\theta(1_M, \Lambda') = 0$ (Proposition (0.8) (ii) of [B2]). As $ks(V_1) \equiv ks(V_2)$, this implies there is a homeomorphism $F : V_1 \to V_2$ extending 1_M (see Theorem (0.7) of [B2]). If $\mathscr{L}$ is even, the spin structure on M from V_1 equals that from V_2 and thus $\theta(1_M, \Lambda) \equiv 0$ (Proposition (4.1) (v) of [B2]). As before, this suffices to prove uniqueness.

§3. Realization of homology classes

We prove the existence part of Theorem F in this section. Throughout, W will denote a closed, 1-connected 4-manifold, $\xi_0 \in H_2(W)$ a primitive class and $\upsilon \in H_2(W)$ a class satisfying $\upsilon \cdot \xi_0 = 1$.

Let $E = \{\xi \in H_2(W) \mid \xi \cdot \xi_0 = 0\}$ and define $\mathscr{L}$ to be the restriction of the intersection pairing to E. Set $r = \xi_0 \cdot \xi_0$.

(3.1) LEMMA. *$(E, \mathscr{L})$ presents $H_*(-L(r, 1))$ in a canonical fashion.*

Proof. Now υ defines an element of E^* in an obvious way: $\upsilon_E(\xi) = \upsilon \cdot \xi$. We shall show that

(i) coker $(ad(\mathscr{L})) \cong \mathbb{Z}/r$ generated by υ_E;

(ii) if $r > 0$, $\mathscr{L}^*(\upsilon_E, \upsilon_E) = -1/r \pmod{\mathbb{Z}}$.

Assuming these two claims have been demonstrated, we may readily define the desired presentation

$$0 \longrightarrow H_2(-L(r, 1)) \xrightarrow{h} E \xrightarrow{\mathrm{ad}(\mathscr{L})} E^* \xrightarrow{\partial} H_1(-L(r, 1)) \longrightarrow 0$$

as follows. Let $D(S^2, r)$ be the 2-disk bundle over S^2 with euler number r. It is well-known that $\partial D(S^2, r) = L(r, 1)$ thus giving $L(r, 1)$ the structure of an S^1-bundle over S^2 and further, that any two such fibrings of $L(r, 1)$ are isotopic. Thus we may define a natural isomorphism coker $(ad(\mathscr{L})) \xrightarrow{\cong} H_1(-L(r, 1))$ by sending the class of υ_E in coker $(ad(\mathscr{L}))$ to the negative of the class in $H_1(-L(r, 1))$ carried by a circle fibre. Note that this isomorphism is independent of the choice of υ. Let $\partial : E^* \to H_1(-L(r, 1))$ be the composition

$$E^* \longrightarrow E^*/ad(\mathscr{L})(E) \xrightarrow{\cong} H_1(-L(r, 1)).$$

Now we define $h : H_2(-L(r, 1)) \to E$. When $r \neq 0$, $H_2(-L(r, 1)) = 0$ and so we take $h = 0$. When $r = 0$ we note that $\xi_0 \in E$ and we define $h(\beta) = \xi_0$ where $\beta \in H_2(-L(0, 1)) \cong \mathbb{Z}$ is the generator satisfying $\beta \cdot \partial(\upsilon_E) = 1$. It is an easy exercise to verify that these definitions determine a presentation of $H_*(-L(r, 1))$.

Now we justify the two claims. First we show that υ_E generates coker $(ad(\mathscr{L}))$. To that end let $\rho \in E^*$. As E is a summand of $H_2(W)$, ρ extends to an element of $H_2(W)^*$. By duality we can find a class $\gamma \in H_2(W)$ such that $\gamma \cdot \xi = \rho(\xi)$ for each $\xi \in E$. But then since $\gamma - (\gamma \cdot \xi_0)\upsilon$ is an element of E, we see

$$\rho = (\gamma \cdot \xi_0)\upsilon_E + ad(\mathscr{L})(\gamma - (\gamma \cdot \xi_0)\upsilon).$$

Hence $\operatorname{coker}(ad(\mathscr{L}))$ is generated by υ_E. To see that υ_E has order r in $\operatorname{coker}(ad(\mathscr{L}))$, first observe that for each $\xi \in E$, $(r\upsilon) \cdot \xi = ad(\mathscr{L})(r\upsilon - \xi_0)(\xi)$. Hence $r\upsilon_E \in ad(\mathscr{L})(E)$. On the other hand, if $0 < m < |r|$ and $m\upsilon_E = ad(\mathscr{L})(\xi)$ for some $\xi \in E$ then $m \equiv (m\upsilon) \cdot (\xi_0 - r\upsilon) \equiv \xi \cdot (\xi_0 - r\upsilon) \equiv 0 \pmod r$, a contradiction. Hence υ_E has order precisely $|r|$ in $\operatorname{coker}(ad(\mathscr{L}))$.

Finally suppose $r \neq 0$. We have just observed that $r\upsilon_E = ad(\mathscr{L})(r\upsilon - \xi_0)$. Hence

$$\begin{aligned}
\mathscr{L}^*(\upsilon_E, \upsilon_E) &= \left(\frac{1}{r^2}\right) \mathscr{L}(r\upsilon - \xi_0, r\upsilon - \xi_0) \\
&= \left(\frac{1}{r^2}\right) (r^2\upsilon \cdot \upsilon - 2r + r) \\
&= -\frac{1}{r} \quad (\operatorname{mod} \mathbb{Z}).
\end{aligned}$$

This completes the proof of Lemma (3.1). □

If T_0 is an unknotted torus in S^4, say a torus coming from a Heegard splitting of $S^3 \subseteq S^4$, and if F is a surface in W with 1-connected complement which represents $\xi_0 \in H_2(W)$, then $F \# T_0 \subseteq W \# S^4 = W$ also has a 1-connected complement and represents ξ_0. Hence, for the existence part of Theorem F, it suffices to do the case $g = 0$ when $\Theta(\xi_0) \equiv 0 \pmod 2$, and the case $g = 1$ when $\Theta(\xi_0) \equiv 1 \pmod 2$.

Existence when $\mathscr{L}$ is an odd pairing. Mark the presentation of $H_*(-L(r, 1))$ with $ks(W) \in \mathbb{Z}/2$. According to Theorem A, this marked presentation is realized by a compact, 1-connected 4-manifold V. Define a closed 1-connected 4-manifold W' by $W' = V \cup_f D(S^2, r)$ where the gluing homeomorphism $f : \partial V \to D(S^2, r)$ is just the identity of $L(r, 1)$ when $r \neq 0$. When $r = 0$, f will be either the identity or the homeomorphism which fixes the S^1-fibre and acts as the identity on $H_*(L(0, 1))$, but which switches the two spin structures on $L(0, 1)$. We shall specify which below.

Our goal now is to show that W' is homeomorphic to W by a homeomorphism sending the class ξ_0', carried by the base $S^2 \subseteq D(S^2, r) \subseteq W'$, to ξ_0. This will complete the proof of existence when $\mathscr{L}$ is odd. Now $ks(W') \equiv ks(V) \equiv ks(W)$ and so according to Theorem (1.5) of [F] and its addendum, it suffices to produce an isometry $\Gamma : (H_2(W), \cdot) \to (H_2(W'), \cdot)$ such that $\Gamma(\xi_0) = \xi_0'$.

As V realizes the given presentation, there is an isometry $\Lambda : P \to P_V$ such that $\partial(\Lambda) = 1_{H_*}(-L(r, 1))$. Continuing with the notation from Lemma (3.1), let $\eta = (\Lambda^*)^{-1}(\upsilon_E) \in H_2(V, \partial V)$. Then $\partial\eta = \partial\upsilon_E \in H_1(-L(r, 1))$ is, by construction, represented by a negatively oriented S^1-fibre. Hence there is a 2-chain in V representing η whose boundary is this S^1-fibre. Define $\upsilon' \in H_2(W')$ as the class obtained from the sum of this 2-chain with an appropriate D^2-fibre in $D(S^2, r)$. In the case that $r = 0$ we must take more care. First we choose the gluing map f in the definition of W' so that $\upsilon' \cdot \upsilon' \equiv \upsilon \cdot \upsilon \pmod 2$. Next we replace υ' by $\upsilon' + \frac{1}{2}(\upsilon \cdot \upsilon - \upsilon' \cdot \upsilon')\xi_0$ so that now $\upsilon' \cdot \upsilon' = \upsilon \cdot \upsilon$. These choices will be assumed below. Note that in any event $\upsilon' \cdot \xi'_0 = 1$.

Now it is not hard to see that the inclusions induce isomorphisms $\psi : E \oplus \mathbb{Z}\upsilon \to H_2(W)$ and $\psi' : H_2(V) \oplus \mathbb{Z}\upsilon' \to H_2(W')$. Further there is an obvious isomorphism $\varnothing : E \oplus \mathbb{Z}\upsilon \to H_2(V) \oplus \mathbb{Z}\upsilon'$ which restricts to Λ on E and sends υ to υ'. Set $\Gamma = \psi' \cdot \varnothing \cdot \psi^{-1} : H_2(W) \to H_2(W')$. Clearly $\Gamma(\xi_1) \cdot \Gamma(\xi_2) = \xi_1 \cdot \xi_2$ for each $\xi_1, \xi_2 \in E$. We also have that when $\xi \in E$,

$$\begin{aligned} \Gamma(\upsilon) \cdot \Gamma(\xi) &= \upsilon' \cdot \Gamma(\xi) \\ &= \eta \cdot \Lambda(\xi) \quad \text{by construction} \\ &= \Lambda^*(\eta) \cdot \xi \\ &= \upsilon \cdot \xi \quad \text{by definition of } \eta. \end{aligned}$$

It follows that

$$\Gamma(\kappa) \cdot \Gamma(\xi) = \kappa \cdot \xi \tag{3.2}$$

for each $\kappa \in H_2(W)$ and each $\xi \in E$. We will be finished when we indicate why $\upsilon' \cdot \upsilon' = \upsilon \cdot \upsilon$ (implying Γ is an isometry) and why $\Gamma(\xi_0) = \xi'_0$. To prove the latter, we use (3.2) to deduce that if $\xi' \in H_2(V)$, $\Gamma(\xi_0) \cdot \xi' = \xi_0 \cdot \Gamma^{-1}(\xi')$. But $\Gamma^{-1}(\xi') \in E$ and thus $\Gamma(\xi_0) \cdot \xi' = 0$ for each $\xi' \in H_2(V)$. It can be shown that a class in $H_2(W')$ which kills $H_2(V)$, $\Gamma(\xi_0)$ for instance, must be a multiple of ξ'_0. We also know that ξ_0, and thus $\Gamma(\xi_0)$, is primitive. Hence $\Gamma(\xi_0) = \pm\xi'_0$. To show the sign is $+1$ we consider, first of all, the case $r = 0$. Then $\xi_0 \in E$ and so by (3.2), $\upsilon' \cdot \Gamma(\xi_0) = \Gamma(\upsilon) \cdot \Gamma(\xi_0) = \upsilon \cdot \xi_0 = 1$. But $\upsilon' \cdot \xi'_0 = 1$ by construction and therefore $\Gamma(\xi_0) = \xi'_0$. If $r \neq 0$, recall $\xi_0 - r\upsilon \in E$. Then $\Gamma(\xi_0 - r\upsilon) \in H_2(V)$ implies $0 = \xi'_0 \cdot \Gamma(\xi_0 - r\upsilon) = \xi'_0 \cdot \Gamma(\xi_0) - r$. Hence $\xi'_0 \cdot \Gamma(\xi_0) = r$ and again we must have $\Gamma(\xi_0) = \xi'_0$.

We may now complete the case where $\mathscr{L}$ is odd by explaining why $\upsilon' \cdot \upsilon' = \upsilon \cdot \upsilon$. When $r = 0$ this was by construction. Assume then that $r \neq 0$. Now $H_1(-L(r, 1)) \cong \mathbb{Z}/r$ and thus it can be shown $r\upsilon = \xi_0 + \xi$ for some $\xi \in E$. Then $r\upsilon' = \Gamma(r\upsilon) = \xi'_0 + \Lambda(\xi)$. Hence

$$r^2\upsilon' \cdot \upsilon' = \xi'_0 \cdot \xi'_0 + \Lambda(\xi) \cdot \Lambda(\xi) = \xi_0 \cdot \xi_0 + \xi \cdot \xi = r^2\upsilon \cdot \upsilon,$$

and we obtain the desired identity.

Remark. With a little more work, one can arrange for the gluing map to be the identity in the $r=0$ case. For if $\upsilon' \cdot \upsilon' \not\equiv \upsilon \cdot \upsilon \pmod 2$, we use Proposition (0.8) (ii) of [B2] to select a morphism $(1_{\partial V}, \Lambda_1)$ of V such that the obstruction class $\theta(1_{\partial V}, \Lambda_1)$ evaluates nontrivially on an S^1-fibre. If we now replace Λ by $\Lambda_1 \circ \Lambda$ and construct υ' with respect to the new Λ, we shall have $\upsilon' \cdot \upsilon' \equiv \upsilon \cdot \upsilon \pmod 2$.

Existence when $\mathscr{L}$ is an even pairing and $\Theta(\xi_0) \equiv 0$. Let P be the presentation of $H_*(-L(r, 1))$ determined by Lemma (3.1). If $r \neq 0$, $\mathrm{Spin}_P(-L(r, 1))$ has a unique element σ and we mark P with it. If $r=0$, $\mathrm{Spin}_P(-L(r, 1)) = \mathrm{Spin}(-L(r, 1))$ has two elements. Mark P with the unique spin structure extending over $D(S^2, 0)$ if W is even and the other one if W is odd.

Realize this marked presentation by a compact, 1-connected 4-manifold V and construct W' as before, though now we shall take the identity as our gluing map in all cases. The previous argument produces an isometry $\Gamma : (H_2(W), \cdot) \to (H_2(W'), \cdot)$ with $\Gamma(\xi_0) = \xi_0'$. The only point to verify is that when $r=0$, the congruency $\upsilon' \cdot \upsilon' \equiv \upsilon \cdot \upsilon$ holds. But as $\mathscr{L}$ is even, W (respectively W') is spin if and only if $\upsilon \cdot \upsilon$ (respectively $\upsilon' \cdot \upsilon'$) $\equiv 0 \pmod 2$. Now $\sigma \in \mathrm{Spin}_P(-L(0, 1))$ was selected so that W is spin if and only if W' is. Hence $\upsilon' \cdot \upsilon' \equiv \upsilon \cdot \upsilon \pmod 2$.

If $(H_2(W), \cdot)$ is even, the existence of Γ is enough to deduce the existence of a homeomorphism $f : W' \to W$ with $f_*(\xi_0') = \xi_0$ and so we are done in this case. The final case for consideration is when $(H_2(W), \cdot)$ is odd. This is precisely the case where ξ_0 is characteristic and so by the hypothesis $\Theta(\xi_0) \equiv 0 \pmod 2$, we see

$$\begin{aligned} ks(W) &\equiv \tfrac{1}{8}[\mathrm{signature}(W) - \xi_0 \cdot \xi_0] \pmod 2 \\ &\equiv \tfrac{1}{8}[\mathrm{signature}(W) - r]. \pmod 2 \end{aligned}$$

On the other hand

$$\begin{aligned} ks(W') &\equiv ks(V) \pmod 2 \\ &\equiv \tfrac{1}{8}[\mathrm{signature}(V) - \mu(-L(r, 1); \sigma)] \pmod 2. \end{aligned}$$

Now Novikov additivity shows signature (V) = signature (W) − sign (r). We also know that as $(H_2(W'), \cdot)$ is odd, the spin structure σ does not extend over $D(S^2, r)$. This determines $\sigma \in \mathrm{Spin}(-L(r, 1))$ and it may be shown $\mu(-L(r, 1); \sigma) = \mathrm{sign}(r) - r$ (Theorem (6.5) of [KT]). Hence

$$ks(W') \equiv \tfrac{1}{8}[\mathrm{signature}(W) - r] \equiv ks(W).$$

We conclude that Γ is realized by a homeomorphism.

Existence when $\mathscr{L}$ is even and $\Theta(\xi_0) \equiv 1$. Let $T = S^1 \times S^1$ and denote by $D(T, r)$ the 2-disk bundle over T with euler number r. Set $M_r = -\partial D(T, r)$. Consider the bilinear pairing $(\hat{E}, \hat{\mathscr{L}}) = (E, \mathscr{L}) \oplus (\mathbb{Z}^2, 0)$. Using Lemma (3.1), it is not hard to see that $(\hat{E}, \hat{\mathscr{L}})$ presents $H_*(M_r)$ in such a way that if $v_{\hat{E}}$ is the extension, by zero, of v_E to $\hat{E}$, then $\partial v_{\hat{E}} \in H_1(M_r)$ corresponds to a negatively oriented S^1-fibre in M_r. Fix such a presentation P and mark it with $\sigma \in \mathrm{Spin}_P(M_r)$ satisfying

$$\begin{cases} \text{(i) } ks(W) \equiv \frac{1}{8}[\text{signature}(\mathscr{L}) - \mu(M_r; \sigma)] \pmod 2; \\ \text{(ii) } \sigma \text{ does not extend across } D(T, r). \end{cases} \tag{3.3}$$

It is not obvious at first glance that there are such spin structures. We shall prove this in a moment. Assuming it, realize (P, σ) by V and form $W' = V \cup D(T, r)$, glued with the identity function. As $\mathscr{L}$ is even but W is odd, $v \cdot v \equiv 1 \pmod 2$. The choice of σ shows that W' is also odd and so $v' \cdot v' \equiv 1 \pmod 2$ also. The construction of an isometry $\Gamma : H_2(W) \to H_2(W')$ may now proceed as before. Finally, condition (i) on σ shows

$$\begin{aligned} ks(W') \equiv ks(V) &\equiv \tfrac{1}{8}[\text{signature}(\mathscr{L}) - \mu(M_r; \sigma)] \\ &\equiv ks(W) \quad \text{by (3.3) (i).} \end{aligned}$$

Hence Γ is realised by a homeomorphism and this final case of existence for Theorem F will be done when we explain how to find $\sigma \in \mathrm{Spin}_P(M_r)$ satisfying (i) and (ii) above.

Let $\mathrm{Spin}^0(M_r)$ denote the collection of spin structures on M_r which do not extend over $D(T, r)$. It is not hard to see that $\mathrm{Spin}^0(M_r)$ has four elements, each pair differing by an element $p^*(H^1(T; \mathbb{Z}/2)) \subseteq I^1(M_r)$, where $p : M_r \to T$ is the projection.

(3.4) LEMMA. *If $r \neq 0$, $\mathrm{Spin}^0(M_r) = \mathrm{Spin}_P(M_r)$.*

Proof. If r is odd, both sets equal $\mathrm{Spin}(M_r)$, so we may assume r is even.

Now $\mathrm{Spin}_P(M_r)$ is the $I^1(M_r)$-orbit in $\mathrm{Spin}(M_r)$ of spin structures whose associated quadratic enhancement of the link pairing equals q_P, the one defined by the presentation P (see Proposition (2.11) of [B2] for instance). A calculation similar to that of $\mathscr{L}^*(v_E, v_E)$ in Lemma (3.1) shows $q_P(\partial v_{\hat{E}}) \equiv (2r)^{-1} - \frac{1}{2} \pmod{\mathbb{Z}}$. On the other hand, using the fact that $r \neq 0$, the remarks prior to this lemma imply that $\mathrm{Spin}^0(M_r)$ is an $I^1(M_r)$ orbit, whose complement consist of those spin structures extending over $D(T, r)$. Now $H_1(D(T, r))$ is free and so the quadratic enhancement, q say, of a spin structure extending over $D(T, r)$ may be calculated from the homology sequence of the pair $(D(T, r), M_r)$. In particular we have $q(\partial v_{\hat{E}}) \equiv (2r)^{-1} \pmod{\mathbb{Z}}$. Thus $q \neq q_P$. The lemma now follows. □

As $\mathrm{Spin}_P(M_r) = \mathrm{Spin}(M_r)$ when $r = 0$, $\mathrm{Spin}^0(M_r) \subseteq \mathrm{Spin}_P(M_r)$ always. We need to produce an element $\sigma \in \mathrm{Spin}^0(M_r)$ satisfying (i). To do this, let $\chi : \mathrm{Spin}^0(M_r) \to \mathrm{Spin}(T)$ be the function of Lemma (6.2) of [KT]. It satisfies $\chi(p^*(x) \cdot \sigma) = x \cdot \chi(\sigma)$ for each $x \in H^1(T; \mathbb{Z}/2)$. Let $\beta : \Omega_2^{\mathrm{Spin}} \to \mathbb{Z}/2$ be the isomorphism. Now T admits four spin structures, precisely three of which are nullcobordant. Hence, using Theorem (6.5) of [KT], if $\sigma \in \mathrm{Spin}^0(M_r)$ there is an $x \in H^1(T; \mathbb{Z}/2)$ such that

$$\begin{aligned}\mu(M_r; p^*(x) \cdot \sigma) - \mu(M_r; \sigma) &\equiv 8 \cdot \beta(T^{x \cdot \chi(\sigma)}) - 8 \cdot \beta(T^{\chi(\sigma)}) \\ &\equiv 8 \pmod{16}.\end{aligned}$$

If σ does not satisfy (i), then $p^*(x) \cdot \sigma \in \mathrm{Spin}^0(M_r)$ will. This completes the proof of the existence part of Theorem F.

(3.5) *Remark.* Suppose ξ_0 is characteristic and let F be a surface of genus $g \geq 1$. We can use the method above to construct directly a locally-flat embedding of F in W with 1-connected complement which realizes ξ_0. We let $D(F, r)$ denote the 2-disk bundle over F of euler number r and set $M_r = -\partial D(F, r)$. Then we show $(\hat{E}, \hat{\mathscr{L}}) = (E, \mathscr{L}) \oplus (\mathbb{Z}^{2g}, 0)$ presents $H_*(M_r)$ appropriately, say by a presentation P. Now P must be marked by a spin structure $\sigma \in \mathrm{Spin}_P(M_r)$ satisfying the conditions (3.3). As $g \geq 1$, this can always be done. Indeed there will be $2^g + (-1)^{\Theta(\xi_0)} 2^{g-1}$ such markings. To see this let $\mathrm{Spin}^0(M_r)$ be those spin structures on M_r not extending across $D(F, r)$. As before, $\mathrm{Spin}^0(M_r) \subseteq \mathrm{Spin}_P(M_r)$. Lemma (6.2) of [KT] produces a bijection $\chi : \mathrm{Spin}^0(M_r) \to \mathrm{Spin}(F)$ and using Theorem (6.5) of that paper we can show that for $\sigma \in \mathrm{Spin}^0(M_r)$ condition (3.3) (i) is satisfied if and only if $\Theta(\xi_0) \equiv \beta(F^{\chi(\sigma)}) \pmod 2$. Now there are $2^g + 2^{g-1}$ nullcobordant spin structures on F and $2^g - 2^{g-1}$ non-nullcobordant ones. The result follows.

§4. Uniqueness

In this section we complete the proof of Theorem F and prove Theorem G.

Suppose F_1, and F_2 are two locally flat surfaces of genus g in W with 1-connected complements which represent ξ_0. Let $h : F_1 \to F_2$ be a homeomorphism. According to §9.3 of [FQ], each has a 2-disc bundle tubular neighbourhood, say $D(F_1)$ and $D(F_2)$. As the euler number of these bundles are each equal to $\xi_0 \cdot \xi_0 = r$, there is a homeomorphism $H : D(F_1) \to D(F_2)$ extending h. We would like to extend H to a homeomorphism $\hat{H} : W \to W$ isotopic to the identity. We shall see that this is always possible as long as ξ_0 is not characteristic. If ξ_0 is characteristic, there will be an obstruction, but it may be nullified by appropriately replacing the homeomorphism $h : F_1 \to F_2$. Now the details.

Let $M_1 = -\partial D(F_1)$ and $M_2 = -\partial D(F_2)$. Consider the exteriors of F_1 and $F_2 : V_1 = W \backslash \mathring{D}(F_1)$ and $V_2 = W \backslash \mathring{D}(F_2)$. By hypothesis these are compact, 1-connected 4-manifolds whose boundaries are homeomorphic via $f = H \mid M_1 : M_1 \to M_2$. Note that f takes a positively oriented S^1-fibre in M_1 to a positively oriented S^1-fibre in M_2.

Denote by $\varphi_j : H_2(V_j) \to H_2(W)$ the homomorphism induced by the inclusion $(j = 1, 2)$. Recall $E = \{\xi \in H_2(W) \mid \xi \cdot \xi_0 = 0\}$.

(4.1) LEMMA. Image $(\varphi_j) = E$ *for* $j = 1, 2$.

Proof. Fix j. From the Thom isomorphism theorem, the natural homomorphism $H_2(W) \to H_2(W, V_j) \cong H_2(D(F_j), M_j) \cong \mathbb{Z}$ sends a class $\xi \in H_2(W)$ to $\xi \cdot \xi_0$. Hence its kernel is E. But this kernel is clearly image (φ_j). $\square$

It follows from Lemma (4.1) that for each j, there is a sequence $0 \to H_3(W, V_j) \to H_2(V_j) \to E \to 0$. As E is free abelian, these sequences split, and thus there is an isomorphism $\Lambda : H_2(V_1) \to H_2(V_2)$ making the following diagram commute,

$$\begin{tikzcd}
0 \arrow[r] & H_3(W, V_1) \arrow[rr] & & H_2(V_1) \arrow[r] \arrow[ddd, "\Lambda"] & E \arrow[r] \arrow[ddd, "1_E"] & 0 \\
& H_3(D(F_1), M_1) \arrow[r] \arrow[u, "\cong"] \arrow[d, "H_*"] & H_2(M_1) \arrow[ur, "h_1"] \arrow[d, "f_*"] & & & \\
& H_3(D(F_2), M_2) \arrow[r] \arrow[d, "\cong"] & H_2(M_2) \arrow[dr, "h_2"] & & & \\
0 \arrow[r] & H_3(W, V_2) \arrow[rr] & & H_2(V_2) \arrow[r] & E \arrow[r] & 0
\end{tikzcd} \tag{4.2}$$

It is not hard to verify that any such isomorphism Λ is an isometry. Our next goal is to show that Λ may be chosen so that $\partial(\Lambda) = f_*$ with respect to the presentations P_{V_1} and P_{V_2}.

For each $j = 1, 2$, let $\kappa_j : H_2(W) \to H_2(V_j, M_j)$ be the composite $H_2(W) \to H_2(W, D(F_j)) \cong H_2(V_j, M_j)$.

(4.3) LEMMA. *The following diagram commutes*

$$\begin{tikzcd}
H_2(V_1) \arrow[rr] \arrow[dr, "\varphi_1"] \arrow[dd, "\Lambda"'] & & H_2(V_1, M_1) \\
& H_2(W) \arrow[ur, "\kappa_1"'] \arrow[dr, "\kappa_2"] & \\
H_2(V_2) \arrow[rr] \arrow[ur, "\varphi_2"'] & & H_2(V_2, M_2) \arrow[uu, "\Lambda^*"']
\end{tikzcd}$$

Proof. We only need to verify that $\kappa_1 = \Lambda^* \circ \kappa_2$. To see that let $\mu \in H_2(W)$ and $\xi \in H_2(V_1)$ be arbitrary. Then

$$\begin{aligned}
\Lambda^*(\kappa_2(\mu)) \cdot \xi &= \kappa_2(\mu) \cdot \Lambda(\xi) \\
&= \mu \cdot \varphi_2(\Lambda(\xi)) \\
&= \mu \cdot \varphi_1(\xi) \\
&= \kappa_1(\mu) \cdot \xi.
\end{aligned}$$

It follows that $\Lambda^* \circ \kappa_2 = \kappa_1$. $\square$

(4.4) LEMMA. *There is an isometry* $\Lambda : H_2(V_1) \to H_2(V_2)$ *making diagram* (4.2) *commute such that* $\partial(\Lambda) = f_*$.

Proof. Examination of diagram (4.2) shows that $\partial(\Lambda)_2 = f_* : H_2(M_1) \to H_2(M_2)$. Now both $\partial(\Lambda)$ and f_* preserve the intersection pairings $H_1(M_j) \times H_2(M_j) \to \mathbb{Z}$. By the nonsingularity of these pairings (at least after dividing $H_1(M_j)$ by its torsion subgroup) we see that $\partial(\Lambda)_1$ and $f_* : H_1(M_1) \to H_1(M_2)$ are equal up to torsion. When $r = 0$, both $H_1(M_1)$ and $H_1(M_2)$ are free and so $\partial(\Lambda) = f_*$.

Assume now that $r \neq 0$. Recall $\upsilon \in H_2(W)$ satisfies $\upsilon \cdot \xi_0 = 1$. Then if $\partial_J : H_2(V_j, M_j) \to H_1(M_j)$ is the boundary homomorphism, $(\partial_j \circ \kappa_j)(\upsilon) \in H_1(M_j)$ is represented by a negatively oriented S^1-fibre in M_j. Hence, by construction of $f : M_1 \to M_2$,

$$\begin{aligned}
f_*(\partial_1(\kappa_1(\upsilon))) = \partial_2(\kappa_2(\upsilon)) &= \partial_2 \circ (\Lambda^*)^{-1}(\kappa_1(\upsilon)) \quad \text{by (4.3)} \\
&= \partial(\Lambda)_1(\partial_1(\kappa_1(\upsilon))).
\end{aligned}$$

Now as $r \neq 0$, $T_1(M_j) \cong \mathbb{Z}/r$ and is generated by $\partial_j(\kappa_j(\upsilon))$ $(j = 1, 2)$. Hence we have shown $\partial(\Lambda)_1 \mid T_1(M) = f_* \mid T_1(M_1)$. But we observed that f_* and $\partial(\Lambda)_1$ differ by torsion elements of $H_1(M_2)$. It follows that $\partial(\Lambda)_1 = f_* \mid H_1(M) + \psi$ for some homomorphism $\psi : (H_1(M_1), T_1(M_1)) \to (T_1(M_2), 0)$. We now proceed as in the proof of Proposition (1.6) of [B2] to show how Λ may be altered to satisfy $\partial(\Lambda) = f_*$ and to make diagram (4.2) commute (see pages 338–339 of [B2]). □

If $\Lambda : H_2(V_1) \to H_2(V_2)$ is chosen to satisfy the conclusion of Lemma (4.4) then $(f, \Lambda) : V_1 \to V_2$ is a morphism. Recall $\theta(f, \Lambda)$ the associated obstruction class in $I^1(M_2)$ ([B2]).

(4.5) LEMMA. *If* $\theta(f, \Lambda) = 0$, *then* H *extends to a homeomorphism* $\hat{H} : W \to W$ *which is isotopic to the identity. In particular, the inclusion* $K : D(F_1) \subseteq W$ *is ambient isotopic to* $H \circ K$.

Proof. Now $ks(V_1) \equiv ks(W) \equiv ks(V_2)$ and so the hypothesis $\theta(f, \Lambda) = 0$ implies that f extends to a homeomorphism $H' = V_1 \to V_2$ with $H'_* = \Lambda$ (see Theorem (0.7) of [B2]). Let $\hat{H} = H \cup H' : W = D(F_1) \cup V_1 \to W = D(F_2) \cup V_2$. Then $\hat{H}$ is clearly a homeomorphism for which $\hat{H}_*(\xi_0) = \xi_0$. From diagram (4.2) we see that if $\xi \in H_2(V_1)$ then

$$\hat{H}_*(\varphi_1(\xi)) = \varphi_2(H'_*(\xi)) = \varphi_2(\Lambda(\xi)) = \varphi_1(\xi).$$

Hence $\hat{H}_* \mid E = 1_E$. Now if $r \neq 0$, ξ_0 and E span $H_2(W)$ rationally. Thus in this case, $\hat{H}_* = 1_{H_2(W)}$. According to [Q], $\hat{H}$ is isotopic to the identity.

Assume now that $r = 0$ and recall $\upsilon \in H_2(W)$ satisfies $\upsilon \cdot \xi_0 = 1$. As noted above, $\hat{H}_* \mid E = 1_E$ and thus $\hat{H}_*(\xi) \cdot \upsilon = \xi \cdot \upsilon$ for each $\xi \in E$. This means that $\hat{H}_*(\upsilon) = \upsilon + m\xi_0$ for some $m \in \mathbb{Z}$. Then $\upsilon \cdot \upsilon = \hat{H}_*(\upsilon) \cdot \hat{H}_*(\upsilon) = \upsilon \cdot \upsilon + 2m$. Then $m = 0$ and so $\hat{H}_*(\upsilon) = \upsilon$. As υ and E span $H_2(W)$, we conclude $\hat{H}_* = 1_{H_2(W)}$ in the case $r = 0$ also. As above, this means $\hat{H}$ is isotopic to the identity. □

(4.6) *Remark.* We note the following consequence of the last lemma. Suppose F_1 and F_2 are two locally flat surfaces in W with 1-connected complements which represent ξ_0. Let $h : F_1 \to F_2 \subseteq W$ be a homeomorphism. Then h is isotopic to the inclusion $F_1 \subseteq W$ if and only if for the restriction $f : M_1 \to M_2$ of some homeomorphism $H : D(F_1) \to D(F_2)$ extending h, there is a morphism $(f, \Lambda) : H_2(V_1) \to H_2(V_2)$ such that Λ makes diagram (4.2) commute and $\theta(f, \Lambda) = 0$.

Recall from the proof of Lemma (4.4) that the class $\partial_2(\kappa_2(\upsilon))$ is represented by a negatively oriented S^1-fibre in M_2.

(4.7) LEMMA. *For any isometry* $\Lambda : H_2(V_1) \to H_2(V_2)$ *making diagram* (4.2) *commute and with* $\partial(\Lambda) = f_*$, $\theta(f, \Lambda)(\partial_2(\kappa_2(\upsilon))) = 0$.

Proof. When r is odd, $\partial_2(\kappa_2(\upsilon))$ has odd order and so the result is clear. Suppose then that r is even. Fix a spin structure $\sigma_1 \in \mathrm{Spin}\,(M_1)$ extending over $D_1(F_1)$. If $\sigma_2 = f_{\#}(\sigma_1)$, it is clear σ_2 extends over $D(F_2)$. Then by Proposition (4.1) of [B2],

$$\begin{aligned}
&\theta(f, \Lambda)(\partial_2(\kappa_2(\upsilon)))\\
&\quad \equiv w_2(V_2, M_2; f_{\#}(\sigma_1))(\kappa_2(\upsilon)) - w_2(V_2, M_2; \pi_\Lambda(\sigma_1))(\kappa_2(\upsilon))\\
&\quad \equiv w_2(V_2, M_2; \sigma_2)(\kappa_2(\upsilon)) - w_2(V_1, M_1, \sigma_1)(\Lambda^*(\kappa_2(\upsilon)))\\
&\quad \equiv w_2(V_2, M_2; \sigma_2)(\kappa_2(\upsilon)) - w_2(V_1, M_1, \sigma_1)(\kappa_1(\upsilon)) \quad \text{by Lemma (4.3)}\\
&\quad \equiv \langle \omega_2(W), \upsilon \rangle - \langle \omega_2(W), \upsilon \rangle\\
&\quad \equiv 0 \pmod 2
\end{aligned}$$

as σ_1 extends over $D(F_1)$ and σ_2 extends over $D(F_2)$. Evidently this completes the proof. □

Proof of uniqueness when $\mathscr{L}$ is odd. Let $\Lambda : H_2(V_1) \to H_2(V_2)$ be an isometry as guaranteed by Lemma (4.4). According to Proposition (0.8) (ii) of [B2], there is a morphism $(f, \Lambda') : V_1 \to V_2$ with $\theta(f, \Lambda') = 0$. Now Λ' can be chosen so that diagram (4.2) still commutes. This is because Λ' may be expressed as $\Lambda' = \Lambda + h_2 \circ \psi \circ ad(\mathscr{L}_1)$ where $\psi : H_2(V_1, M_1) \to H_2(M_2)$ is a certain homomorphism with image generated by a class $\beta \in H_2(M_2)$ whose dual in $H^1(M_2)$ reduces (mod 2) to $\theta(f, \Lambda)$. When $r \neq 0$, $\varphi_2 \circ h_2 = 0$ and so $\varphi_2 \circ \Lambda' = \varphi_2 \circ \Lambda = \varphi_1$, which is what we wanted. When $r = 0$, we can use Lemma (4.7) to choose an appropriate β with $\varphi_2(h_2(\beta)) = 0$. Again we shall have $\varphi_2 \circ \Lambda' = \varphi_1$.

In all cases then, we can arrange for the hypotheses of Lemma (4.5) to hold. Hence the inclusion $K : D(F_1) \to W$ is ambient isotopic to $H \circ K$. As the homeomorphism $h : F_1 \to F_2$ and its extension $H : D(F_1) \to D(F_2)$ were chosen arbitrarily, the uniqueness statement in Theorem F and part (i) of Theorem G hold when $\mathscr{L}$ is odd.

Proof of uniqueness when $\mathscr{L}$ is even. Let σ_1 and σ_2 be the unique spin structures on M_1 and M_2 extending over V_1 and V_2. According to Proposition (4.1) (v) of [B2], if $(f, \Lambda) : V_1 \to V_2$ is any morphism, then $\theta(f, \Lambda)$ is determined by the identity $f_{\#}(\sigma_1) = \theta(f, \Lambda) \cdot \sigma_2$. We must therefore arrange for $f_{\#}(\sigma_1) = \sigma_2$, if possible.

Consider, first of all, the case where W is spin. Then σ_j extends uniquely to a spin structure $\tilde{\sigma}_j$ on $D(F_j)$. Now it may be that $H_{\#}(\tilde{\sigma}_1) \neq \tilde{\sigma}_2$, but this may be corrected without altering h as follows. Let $p : D(F_2) \to F_2$ be the projection and fix a class $x \in H^1(F_2; \mathbb{Z}/2)$ such that $H_{\#}(\tilde{\sigma}_1) = p^*(x) \cdot \tilde{\sigma}_2$. Choose any $\tilde{x} \in H^1(F_2)$ reducing to x and represent $\tilde{x}$ by a function $\gamma : F_2 \to S^1$. Now S^1 acts on $D(F_2)$ by rotation of the D^2-fibres. In particular we may define a homeomorphism $R : (D(F_2), F_2) \to (D(F_2), F_2)$ by $R(z) = \gamma(p(z))(z)$. Clearly $R \mid F_2 = 1_{F_2}$. If we replace

H by $R \circ H$, then we still have an extension of $h : F_1 \to F_2$, but now $H_\#(\tilde{\sigma}_1) = \tilde{\sigma}_2$. Restricting to M_1 and M_2 gives $f_\#(\sigma_1) = \sigma_2$. Thus, as in the case when $\mathscr{L}$ was odd, no matter which homeomorphism $h : F_1 \to F_2$ was chosen, the inclusion $k : F_1 \to W$ is ambient isotopic to $h \circ k$. This gives the uniqueness part of Theorem F when W is spin as well as the portion of Theorem G (ii) referring to $\mathscr{I}$. To complete the proof of Theorem G (ii), we must show that $\tilde{\mathscr{I}}$ has 2^{2g} elements. But our arguments show that when W is spin,

(i) two homeomorphisms $H_1, H_2 : (D(F_1), F_1) \to (D(F_2), F_2)$ are ambiently isotopic if and only if $(H_1)_\#(\tilde{\sigma}_1) = (H_2)_\#(\tilde{\sigma}_1)$;

(ii) for each $x \in H^1(F_2; \mathbb{Z}/2)$, there is a homeomorphism $H_x : (D(F_1), F_1) \to (D(F_2), F_2)$ such that $(H_x)_\#(\tilde{\sigma}_1) = p^*(x) \cdot \tilde{\sigma}_2$.

As $D(F_2)$ admits 2^{2g} spin structures, it follows that $\tilde{\mathscr{I}}$ has 2^{2g} elements.

The final case for consideration is when W is odd and $\mathscr{L}$ is even. This is precisely the case where ξ_0 is characteristic. We shall see that it may not be possible to isotope the inclusion $k : F_1 \to W$ to $h \circ k$, but that there is a homeomorphism $h_1 : F_1 \to F_2$ such that k is isotopic to $h_1 \circ k$.

Let P_j denote the presentation of $H_*(M_j)$ arising the homology sequence of (V_j, M_j) $(j = 1, 2)$. Let $\mathrm{Spin}^0(M_j)$ be the collection of spin structures on M_j not extending over $D(F_j)$. Then $\mathrm{Spin}^0(M_j)$ has 2^{2g} elements and arguing as in Lemma (3.4) it can be shown that $\mathrm{Spin}^0(M_j) = \mathrm{Spin}_{P_j}(M_j)$ if $r \neq 0$ and that $\mathrm{Spin}^0(M_j)$ is exactly one half of $\mathrm{Spin}_{P_j}(M_j) = \mathrm{Spin}(M_j)$ when $r = 0$. Lemma (6.2) of [KT] constructs a bijection $\chi_j : \mathrm{Spin}^0(M_j) \to \mathrm{Spin}(F_j)$ for which it may be shown that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Spin}^0(M_1) & \xrightarrow{f_\#} & \mathrm{Spin}^0(M_2) \\ {\scriptstyle \chi_1}\downarrow & & \downarrow{\scriptstyle \chi_2} \\ \mathrm{Spin}(F_1) & \xrightarrow{h_\#} & \mathrm{Spin}(F_2). \end{array}$$

Thus $f_\#(\sigma_1)$ depends only on h and further the following statements are equivalent:

(i) the inclusion $K : D(F_1) \subseteq W$ is ambiently isotopic to $H \circ K$;

(ii) the inclusion $k : F_1 \subseteq W$ is ambiently isotopic to $h \circ k$;

(iii) $f_\#(\sigma_1) = \sigma_2$;

(iv) $h_\#(\chi_1(\sigma_1)) = \chi_1(\sigma_2)$.

Thus the natural restriction map $\tilde{\mathscr{I}} \to \mathscr{I}$ is bijective when ξ_0 is characteristic. We remark further that $h_\#(\chi_1(\sigma_1))$ is spin cobordant to $\chi_2(\sigma_2)$. This follows from the fact that if $\beta : \Omega_2^{\mathrm{Spin}} \to \mathbb{Z}/2$ is the isomorphism, then

$$\beta(F_2^{h_\#(\chi_1(\sigma_1))}) \equiv \beta(F_1^{\chi_1(\sigma_1)}) \equiv \Theta(\xi_0) \equiv \beta(F_2^{\chi_2(\sigma_2)}) \tag{4.8}$$

(see Remark (3.5)). Now there are exactly $[2^g + (-1)^{\Theta(\xi_0)} 2^{g-1}]$ spin structures on F_2 satisfying (4.8) and they form an orbit in $\mathrm{Spin}(F_2)$ of the $\mathrm{Homeo}^+(F_2)$ action.

Thus, considering the compositions $h' \circ h$ where $h' \in \text{Homeo}^+ (F_2)$, shows that both $\tilde{\mathscr{I}}$ and $\mathscr{I}$ have $[2^g + (-1)^{\Theta(\xi_0)} 2^{g-1}]$ elements and further that there is a homeomorphism $h_1 : F_1 \to F_2$ such that $h_1 \circ k$ is ambiently isotopic to k. This completes the proofs of Theorem F and Theorem G.

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Received August 5, 1991