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Flat exterior Tor algebras and cotangent complexes[1]

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Introduction

Let A be a ring, B and C A-algebras (commutative with unit) and $D = B \otimes_A C$. It is well known [M, Theorem 2.2, p. 225] that $\mathrm{Tor}^A(B, C)$ is a strictly anticommutative graded D-algebra. So we have a homomorphism of graded D-algebras

$$\gamma : \wedge_D \mathrm{Tor}_1^A(B, C) \to \mathrm{Tor}^A(B, C).$$

In [A_2], M. André has introduced for $n \geq 0$ and W a D-module, homology modules $H_n(A, B, C, W)$ generalizing in some way the classical homology functors of André–Quillen $H_n(R, S, -)$ (see [A_1], [Q_2], [Q_3]).

The purpose of this paper is to relate properties of γ and the vanishing of the functors $H_n(A, B, C, -)$, $n \geq 3$. More precisely, our main result is the following

THEOREM 1. *Let A be a ring, B and C A-algebras and $D = B \otimes_A C$. The following conditions are equivalent:*

(1) *The D-module $\mathrm{Tor}_1^A(B, C)$ is flat and the canonical homomorphism*

$$\gamma : \wedge_D \mathrm{Tor}_1^A(B, C) \to \mathrm{Tor}^A(B, C)$$

is an isomorphism.

(2) $H_j(A, B, C, -) = 0$ *for* $j \geq 3$.

This theorem has as a consequence two important results, the first one is already known but the second isn't.

COROLLARY 2. *Let A be a ring, I an ideal of A and $B = A/I$. The following conditions are equivalent:*

[1] Partially supported by Xunta de Galicia, XUGA 20701A92.

(1) *The B-module* I/I^2 *is flat and the canonical homomorphism*

$$\wedge_B I/I^2 \longrightarrow \operatorname{Tor}^A(B, B)$$

is an isomorphism.

(2) $H_j(A, B, -) = 0$ *for* $j \geq 2$.

This result is due to Quillen [Q_2, Theorem 10.3], [Q_3, Theorem 6.13].

COROLLARY 3. *Let A be a ring, I an ideal of A,* $B = A/I$, *and E the Koszul complex associated to an arbitrary set of generators of I. The following conditions are equivalent*:

(1) *The B-module* $H_1(E)$ *is flat and the canonical homomorphism of graded algebras*

$$\wedge_B H_1(E) \longrightarrow H(E)$$

is an isomorphism.

(2) $H_j(A, B, -) = 0$ *for* $j \geq 3$.

Moreover, the following conditions are equivalent:

(1′) *The B-module* $H_1(E)$ *is projective and the canonical homomorphism of graded algebras*

$$\wedge_B H_1(E) \longrightarrow H(E)$$

is an isomorphism.

(2′) $H^j(A, B, -) = 0$ *for* $j \geq 3$.

The proof of Theorem 1 is divided in two parts. In the first one we use an analogue to the fundamental spectral sequence of Quillen [Q_2, Theorem 6.8] to relate the vanishing of $H_n(A, B, C, -)$ with the structure of the homology algebra of a certain derived tensor product $D \overset{L}{\otimes}_Y D$. In the second part we use a spectral sequence

$$E^2_{p,q} = \operatorname{Tor}_p^{H(Y)}(D, D)_q \Rightarrow H(D \overset{L}{\otimes}_Y D)$$

to compare $\operatorname{Tor}^A(B, C)$ with $H(D \overset{L}{\otimes}_Y D)$.

Proofs. First recall the definition of $H_n(A, B, C, W)$. Let A be a ring, B and C two A-algebras, $D = B \otimes_A C$ and W a D-module. Let X be a cofibrant simplicial

A-algebra resolution of B, let $Y = X \otimes_A C$, and let Z be a cofibrant simplicial Y-algebra resolution of D. Then

$$\mathbf{L}_{B-C|A} := \Omega_{Z|Y} \otimes_Z D$$

is a cofibrant simplicial D-module, whose normalization is a chain complex of projective D-modules independent up to homotopy equivalence of the choice of X and Z, and which therefore represents an object unique up to isomorphism of the derived category of the category of D-modules. For a D-module W

$$H_n(A, B, C, W) = H_n(\mathbf{L}_{B-C|A} \otimes_D W).$$

Notice that if J is the simplicial ideal kernel of the surjective canonical homomorphism $Z \otimes_Y D \to D$, then $\mathbf{L}_{B-C|A} = J/J^2$.

The resolution Z can be obtained by the "step by step" construction [A_1, Chap. IX]. So we can assume $Y_n = Z_n$ for $n = 0, 1$ and so

$$H_n(A, B, C, W) = 0 \qquad \text{for } n = 0, 1.$$

For each p, J_p is the ideal generated by the variables of the polynomial D-algebra $(Z \otimes_Y D)_p$. In particular $J_0 = 0$ and J_p is a regular ideal. Quillen's convergence theorem [Q_3, Theorem 6.12] implies $H_p(J^n) = 0$ for $p < n$.

Therefore the spectral sequence resulting from filtering $Z \otimes_Y D$ by the powers of J, is a convergent spectral sequence located in the first quadrant

$$E^2_{p,q} = H_{p+q}(S^q_D \mathbf{L}_{B-C|A}) \Rightarrow H(D \overset{L}{\otimes}_Y D). \tag{I}$$

since $H(Z \otimes_Y D) = H(D \overset{L}{\otimes}_Y D)$ as follows from [Q_1, Theorem 6-(a), p.II.6.8], because the Y_n-algebra Z_n is free for all n.

This spectral sequence is an analogue to Quillen's fundamental spectral sequence.

With the shuffle product $\underline{\otimes}$ [Q_1, p.II.6.6] $Z \otimes_Y D$ with the differential induced by the face operators, is a strictly anticommutative differential graded D-algebra with a system of divided powers. Moreover $Z \otimes_Y D \supset J \supset J^2 \supset \cdots$ is a filtration of $Z \otimes_Y D$ by differential graded ideals. So the spectral sequence is a spectral sequence of bigraded algebras with divided powers.

Since $H_0(\mathbf{L}_{B-C|A}) = H_1(\mathbf{L}_{B-C|A}) = 0$ we have [Q_2, Corollary 7.30] $H_j(S^n_D \mathbf{L}_{B-C|A}) = 0$ if $j < 2n$, i.e., $E^2_{p,q} = 0$ for $p < q$, and there exists a canonical map $\delta\colon \Gamma_D H_2(\mathbf{L}_{B-C|A}) \to H(D \overset{L}{\otimes}_Y D)$ which is the unique homomorphism of graded

D-algebras with divided powers extending the edge isomorphism $H_2(\mathbf{L}_{B-C|A}) = E^2_{1,1} = H_2(D \overset{L}{\otimes}_Y D)$.

Recall now some notation from [Q_2]. For a D-module T, $K(T, n)$ will be the simplicial D-module whose normalization is the complex with T in dimension n and zero in the remaining ones. We have a canonical morphism

$$\mathbf{L}_{B-C|A} \longrightarrow K(H_2(\mathbf{L}_{B-C|A}), 2)$$

which is a 2-equivalence (i.e., it induces isomorphisms in homology in dimensions ≤ 2).

For an object X, cX will be the constant simplicial object with $(cX)_q = X$, and whose faces and degeneracies are the identity map of X.

Finally, Σ is the suspension functor, so that

$$H_{q+1}(\Sigma X) = H_q(X).$$

The proof of the following proposition is analogous to that of Theorem 10.3 of [Q_2]. We give the details for convenience of the reader.

PROPOSITION 4. *The following conditions are equivalent*

(i) $H_j(A, B, C, -) = 0 \quad$ for $j \geq 3$

(ii) The D-module $H_2(D \overset{L}{\otimes}_Y D)$ is flat and the canonical homomorphism

$$\phi : \Gamma_D H_2(D \overset{L}{\otimes}_Y D) \rightarrow H(D \overset{L}{\otimes}_Y D)$$

is an isomorphism

Proof. From the universal coefficient spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^D_p (H_q(\mathbf{L}_{B-C|A}), -) \Rightarrow H(A, B, C, -)$$

we deduce that condition (i) is equivalent to: $H_2(D \overset{L}{\otimes}_Y D)$ is D-flat and

$$\mathbf{L}_{B-C|A} \longrightarrow K(H_2(\mathbf{L}_{B-C|A}), 2)$$

is an n-equivalence for all n.

Note that $K(H_2(\mathbf{L}_{B-C|A}), 2)$ is homotopically equivalent to $\Sigma\Sigma(c(H_2(\mathbf{L}_{B-C|A})))$. Therefore, using [$Q_2$, 7.21] we obtain

$$\begin{aligned} H_{p+q}(S^q_D K(H_2(\mathbf{L}_{B-C|A}), 2)) &= H_{p+q}(S^q_D \Sigma\Sigma(c(H_2(\mathbf{L}_{B-C|A})))) \\ &= H_p(\wedge^q_D \Sigma(c(H_2(\mathbf{L}_{B-C|A})))) \\ &= H_{p-q}(\Gamma^q_D c(H_2(\mathbf{L}_{B-C|A}))) \\ &= \begin{cases} 0 & \text{if } p-q \neq 0 \\ \Gamma^q_D H_2(\mathbf{L}_{B-C|A}) & \text{if } p-q=0. \end{cases} \end{aligned}$$

So, assuming that $\mathbf{L}_{B-C|A} \to K(H_2(\mathbf{L}_{B-C|A}), 2)$ is an n-equivalence, $n \geq 2$, then by [Q_2, 7.3] so are the induced maps of symmetric powers, hence we have

$$\begin{aligned} E^2_{p,q} = H_{p+q}(S^q_D \mathbf{L}_{B-C|A}) &= H_{p+q}(S^q_D K(H_2(\mathbf{L}_{B-C|A}), 2)) \\ &= \begin{cases} 0 & \text{if } p+q \leq n, p \neq q \\ \Gamma^q_D H_2(\mathbf{L}_{B-C|A}) & \text{if } p+q \leq n, p = q. \end{cases} \end{aligned}$$

If (i) holds we can take $n = \infty$ and so

$$E^2_{p,q} = \begin{cases} 0 & \text{if } p \neq q \\ \Gamma^q_D H_2(\mathbf{L}_{B-C|A}) & \text{if } p = q. \end{cases}$$

So the spectral sequence (I) degenerates showing that the edge homomorphism $\delta : \Gamma_D H_2(\mathbf{L}_{B-C|A}) \to H(D \overset{L}{\otimes}_Y D)$ is an isomorphism. Therefore $\phi : \Gamma_D H_2(D \overset{L}{\otimes}_Y D) \to H(D \overset{L}{\otimes}_Y D)$ is an isomorphism.

Now assume that (ii) holds. We will prove by induction on n that $\mathbf{L}_{B-C|A} \to K(H_2(\mathbf{L}_{B-C|A}), 2)$ is an n-equivalence for all n. Assuming $n \geq 2$ and that it is an n-equivalence, to see that it is an $(n+1)$-equivalence we have to prove that $E^2_{n,1} = H_{n+1}(\mathbf{L}_{B-C|A}) = 0$.

Since

$$E^2_{p,q} = \begin{cases} 0 & \text{if } p+q \leq n, p \neq q \\ \Gamma^q_D H_2(\mathbf{L}_{B-C|A}) & \text{if } p+q \leq n, p = q \end{cases}$$

the only possible non zero differential coming from $E^2_{n,1}$ is

$$E^2_{n,1} = E^p_{n,1} \xrightarrow{d^p} E^p_{p,p} = E^2_{p,p} \qquad \text{with } n = 2p.$$

As the edge homomorphism is an isomorphism we have $d^p = 0$. So $E^2_{n,1} = E^\infty_{n,1} = 0$.

Since $H(Y) = \mathrm{Tor}^A(B, C)$, Theorem 1 is a consequence of Proposition 4 and the following general result.

PROPOSITION 5. *Let Y be a simplicial ring and $D = H_0(Y)$.* Then the following conditions are equivalent:

(ii) The D-module $H_2(D \overset{L}{\otimes}_Y D)$ is flat and the canonical homomorphism

$$\phi : \Gamma_D H_2(D \overset{L}{\otimes}_Y D) \longrightarrow H(D \overset{L}{\otimes}_Y D)$$

is an isomorphism.

(iii) The D-module $H_1(Y)$ is flat and the canonical homomorphism

$$\gamma : \wedge_D H_1(Y) \longrightarrow H(Y)$$

is an isomorphism.

Before proceeding to the proof of Proposition 5 we will need some remarks.

Remark 6. By [Q_1, Theorem 6-b), p.II.6.8] there exists a spectral sequence

$$E^2_{p,q} = \mathrm{Tor}_p^{H(Y)}(D, D)_q \Rightarrow H(D \overset{L}{\otimes}_Y D). \tag{II}$$

Since Y is a simplicial ring and D is a simplicial Y-algebra, this spectral sequence is of bigraded algebras with divided powers.

In fact we have the following. Let Y be a simplicial ring and D a simplicial Y-algebra. Then, on the lines of the construction of [Q_1, pp.II.6.13–6.14], it is possible to generalize the "step by step" method to obtain a bisimplicial Y-algebra P and a morphism $P \to D$ such that:

(1) For each j, $P_{*,j}$ is a free simplicial Y_j-algebra resolution of D_j.
(2) For each i, the graded $H(Y)$-algebra $H(P_{i,*})$ is free as an $H(Y)$-module and the induced sequence

$$\cdots \longrightarrow H(P_{2,*}) \longrightarrow H(P_{1,*}) \longrightarrow H(P_{0,*})$$

is a resolution of $H(D)$.

The details of the construction of P are in [B].

Now, if E is another simplicial Y-algebra and $Y \overset{i}{\longrightarrow} Q \overset{p}{\longrightarrow} E$ is a factorization of the canonical morphism $Y \to E$ with i cofibration and p trivial fibration, then we have a bisimplicial Y-algebra

$$M_{i,j} = P_{i,j} \otimes_{Y_j} Q_j.$$

From the two spectral sequences of a double complex, we obtain

$$E^2_{p,q} = \operatorname{Tor}_p^{H(Y)}(H(D), H(E))_q \Rightarrow H(D \overset{L}{\otimes}_Y E).$$

This spectral sequence is of bigraded algebras with divided powers since it comes from a bisimplicial algebra.

Remark 7. Let L be a flat D-module and consider on the bigraded algebra $\wedge_D L \otimes_D \Gamma_D L$ the unique D-derivation of bidegree $(1, -1)$ such that $d(y \otimes 1) = 0$, $d(1 \otimes \gamma_p x) = x \otimes \gamma_{p-1} x$, $x, y \in L$. Then $(\wedge_D L \otimes_D \Gamma^*_D L, d_*)$ is a flat resolution of the $\wedge_D L$-module D: by Lazard's Theorem, we can assume that L is a free D-module of finite type and then by Künneth formula we can take $L = D$. In this case it is clear.

This flat resolution $M_* = \wedge_D L \otimes_D \Gamma^*_D L$ is graded in the following way: $M_{*,i} = \wedge_D^{i-*} L \otimes_D \Gamma^*_D L$. Using this resolution we deduce

$$\operatorname{Tor}_p^{\wedge_D L}(D, D)_q = \begin{cases} 0 & \text{if } p \neq q \\ \Gamma^p_D L & \text{if } p = q. \end{cases}$$

Now we come to the proof of Proposition 5. Consider the spectral sequence of Remark 6

$$E^2_{p,q} = \operatorname{Tor}_p^{H(Y)}(D, D)_q \Rightarrow H_{p+q}(D \overset{L}{\otimes}_Y D).$$

For it, the following hold:

$$E^2_{p,q} = \operatorname{Tor}_p^{H_0(Y)}(D, D) = 0 \qquad \text{if } p > 0$$

$$E^2_{0,q} = (D \otimes_{H(Y)} D)_q = \begin{cases} 0 & \text{if } q > 0 \\ D & \text{if } q = 0 \end{cases} \tag{III}$$

$$E^2_{1,q} = (H_+(Y) \otimes_{H(Y)} D)_q = (H_+(Y)/H_+(Y)^2)_q.$$

In particular we get an edge isomorphism $\alpha_2 : H_2(D \overset{L}{\otimes}_Y D) \to E^2_{1,1}$ and an isomorphism $E^2_{1,1} = H_1(Y)$, which show that the flatness assumptions in (ii) and (iii) are equivalent.

Let $\Lambda = \wedge_D H_1(Y)$ and consider the homomorphism of bigraded D-algebras with divided powers

$$\Gamma_D E^2_{1,1} = \operatorname{Tor}^{\Lambda}(D, D) \xrightarrow{\gamma_{*,*}} \operatorname{Tor}^{H(Y)}(D, D)$$

where the equality is due to Remark 7 and $\gamma_{*,*}$ is induced by γ. Since γ is bijective in degrees $\leq n$, then $\gamma_{p,q}$ is bijective for $q \leq n$, hence in the spectral sequence (II) we have

$$E^2_{p,q} = \begin{cases} 0 & \text{if } p \neq q,\ q \leq n \\ \Gamma^p_D E^2_{1,1} & \text{if } p = q,\ q \leq n \end{cases} \tag{IV}$$

When (iii) holds we can take $n = \infty$ and get an isomorphism of graded D-algebras with divided powers $\beta : \Gamma_D E^2_{1,1} \xrightarrow{\sim} H(D \overset{L}{\otimes}_Y D)$, hence $\phi = \beta \circ \Gamma_D \alpha_2$ is bijective.

Conversely, let (ii) hold. Since γ_n is an isomorphism for $n = 0, 1$, assume by induction that γ_j is bijective for $j \leq n$ and some $n \geq 1$. We have for $p \leq n$ a diagram

$$\begin{array}{ccc} H_{2p}(D \overset{L}{\otimes}_Y D) & \xrightarrow{\alpha_{2p}} & E^2_{p,p} \\ {\scriptstyle \phi_{2p}} \uparrow & & \uparrow {\scriptstyle \psi_{2p}} \\ \Gamma^p_D H_2(D \overset{L}{\otimes}_Y D) & \xrightarrow{\Gamma^p_D \alpha_2} & \Gamma^p_D E^2_{1,1} \end{array}$$

which is commutative because α_{2p} is an edge homomorphism in a spectral sequence (II) of D-algebras with divided powers. Note that ϕ_{2p} is an isomorphism by condition (ii), and ψ_{2p} is an isomorphism by (IV). So α_{2p} is an isomorphism and therefore all differentials of the spectral sequence are zero on $E^r_{p,p}$ when $p \leq n$ and $r \geq 2$. In particular, no differential lands in $E^r_{1,2}$, $E^r_{1,n+1}$, or $E^r_{2,n+1}$ for $n \geq 2$ and $r \geq 2$. Any differential leaving one of these modules lands into some $E^r_{p,*}$ with $p \leq 0$, which is trivial. Thus $E^2_{1,2} = E^\infty_{1,2}$, $E^2_{1,n+1} = E^\infty_{1,n+1}$ and $E^2_{2,n+1} = E^\infty_{2,n+1}$ for $n \geq 2$. We have $E^\infty_{p,q} = 0$ if $p + q$ is odd, and the diagram implies $E^\infty_{p,q} = 0$ when $p \neq q$ and $p + q$ is even $\leq 2n$. Therefore $E^2_{1,2} = E^2_{1,n+1} = E^2_{2,n+1} = 0$ for $n \geq 2$.

By (III) we have $\operatorname{Coker}(\gamma_{n+1}) = E^2_{1,n+1}$, hence γ_{n+1} is surjective. In order to determine $\operatorname{Ker}(\gamma_{n+1})$ we consider an exact sequence of D-modules

$$F''_{n+1} \xrightarrow{\eta} F'_{n+1} \longrightarrow \Lambda_{n+1} \xrightarrow{\gamma_{n+1}} H_{n+1}(Y) \longrightarrow 0$$

in which F''_{n+1} and F'_{n+1} are free. It produces a complex of graded Λ-modules

$$\Lambda \otimes_D F''_{n+1} \xrightarrow{\Lambda \otimes_D \eta} \Lambda \otimes_D F'_{n+1} \longrightarrow \Lambda \xrightarrow{\gamma} H(Y) \longrightarrow 0$$

which is exact in degrees $\leq n + 1$. Thus, by using appropriate graded free Λ-modules G, G', G'' with $G_j = G'_j = G''_j = 0$ for $j \leq n + 1$ we can modify it to obtain the beginning of a graded free resolution of the graded Λ-module $H(Y)$ in the form

$$(\Lambda \otimes_D F''_{n+1}) \oplus G'' \longrightarrow (\Lambda \otimes_D F'_{n+1}) \oplus G' \longrightarrow \Lambda \oplus G \longrightarrow H(Y) \longrightarrow 0.$$

With its help we see that

$$\operatorname{Tor}_1^\Lambda(D, H(Y))_j = \begin{cases} 0 & \text{if } j \le n \\ \operatorname{Coker}(\eta) = \operatorname{Ker}(\gamma_{n+1}) & \text{if } j = n+1. \end{cases}$$

Since γ is surjective in degrees $\le n+1$, the projection $Q = H(Y)/H_1(Y)H(Y) \to H(Y)/H_+(Y) = D$ is bijective in these degrees and thus induces isomorphisms

$$\operatorname{Tor}_2^{H(Y)}(D, Q)_j = \operatorname{Tor}_2^{H(Y)}(D, D)_j = E^2_{2,j} \qquad \text{for } 2 \le j \le n+1.$$

These observations and Remark 7 show that the standard change of rings exact sequence

$$\begin{aligned} \operatorname{Tor}_2^\Lambda(D, D) \longrightarrow \operatorname{Tor}_2^{H(Y)}(D, Q) \longrightarrow (D \otimes_{H(Y)} \operatorname{Tor}_1^\Lambda(D, H(Y))) \\ \longrightarrow \operatorname{Tor}_1^\Lambda(D, D) \longrightarrow \operatorname{Tor}_1^{H(Y)}(D, Q) \longrightarrow 0 \end{aligned}$$

reduces in degree $n+1$ to an exact sequence

$$\operatorname{Tor}_2^\Lambda(D, D)_{n+1} \xrightarrow{\gamma_{2,n+1}} E^2_{2,n+1} \longrightarrow \operatorname{Tor}_1^\Lambda(D, H(Y))_{n+1} \longrightarrow 0.$$

For $n = 1$ the map $\gamma_{2,2}$ is bijective, and for $n > 1$ the module $E^2_{2,n+1}$ is trivial, hence

$$\operatorname{Ker}(\gamma_{n+1}) = \operatorname{Tor}_1^\Lambda(D, H(Y))_{n+1} = 0 \qquad \text{for } n \ge 1.$$

Thus γ_{n+1} is injective, so the induction step is complete and the Proposition is proved.

Corollary 2 follows from Theorem 1 taking $C = B$ and so $D = B$ and $H_n(A, B, C, -) = H_{n-1}(A, B, -)$ [A_2, Example 5].

For Corollary 3, let τ be the set of generators τ_m to which the Koszul complex E is associated. Let R be the free A-algebra with variables t_m and consider the A-algebra homomorphisms $\beta : R \to A$, $\omega : R \to A$, such that $\beta(t_m) = 0$, $\omega(t_m) = \tau_m$. Denote by A_β and A_ω the corresponding R-algebra structures on A. Then there exists an isomorphism of graded B-algebras $H_*(E) = \operatorname{Tor}_*^R(A_\beta, A_\omega)$. Moreover [$A_2$, Example 6], $H_n(R, A_\beta, A_\omega, -) = H_n(A, B, -)$ for $n \ge 3$, and the first part of Corollary 3 follows from Theorem 1.

In order to prove the second part of Corollary 3, we will need some facts about the cotangent complex $\mathbf{L}_{B|A}$. Let X be a simplicial resolution of the A-algebra B obtained by the "step by step" construction [A$_1$, Chap. IX], [A$_2$, p. 327]. In particular consider the first three steps. We begin by choosing a system of generators t_α of the ideal I. The first step is a simplicial A-algebra K with $K_0 = A$, K_1 the polynomial A-algebra on the variables T_α and K_{1+h}, $h > 0$, is the polynomial A-algebra on the variables

$$\sigma_h^{i_1}\sigma_{h-1}^{i_2}\cdots\sigma_1^{i_h}(T_\alpha), \qquad 0 \le i_h < \cdots < i_2 < i_1 \le h,$$

where σ denotes the degeneration operators. The face operators are determined by

$$\varepsilon_1^0(T_\alpha) = 0, \qquad \varepsilon_1^1(T_\alpha) = t_\alpha.$$

In order to construct the second step, we choose representants $s_v \in K_1$ of a set of generators of the B-module

$$\pi_1(K) = \frac{M \cap N}{MN}$$

where M is the ideal of K_1 generated by the elements T_α and N the ideal of K_1 generated by the elements $T_\alpha - t_\alpha$. The second step is a simplicial K-algebra F with $F_0 = K_0$, $F_1 = K_1$, F_2 is the polynomial K_2-algebra on the variables S_v, and F_{2+h}, $h > 0$, is the polynomial K_{2+h}-algebra on the variables

$$\sigma_{1+h}^{i_1}\sigma_h^{i_2}\cdots\sigma_2^{i_h}(S_v), \qquad 0 \le i_h < \cdots < i_2 < i_1 \le 1 + h.$$

The face operators are determined by

$$\varepsilon_2^0(S_v) = 0, \qquad \varepsilon_2^1(S_v) = 0, \qquad \varepsilon_2^2(S_v) = s_v.$$

Similarly the third step is constructed by choosing representants $z_w \in F_2$ of a set of generators of the B-module $\pi_2(F)$ to obtain a simplicial F-algebra G with $G_0 = F_0$, $G_1 = F_1$, $G_2 = F_2$, G_3 is the polynomial F_3-algebra on the variables Z_w and G_{3+h}, $h > 0$, is the polynomial F_{3+h}-algebra on the variables

$$\sigma_{2+h}^{i_1}\sigma_{1+h}^{i_2}\sigma_h^{i_3}\cdots\sigma_3^{i_h}(Z_w), \qquad 0 \le i_h < \cdots < i_2 < i_1 \le 2 + h.$$

The face operators are determined by

$$\varepsilon_3^0(Z_w) = 0, \qquad \varepsilon_3^1(Z_w) = 0, \qquad \varepsilon_3^2(Z_w) = 0, \qquad \varepsilon_3^3(Z_w) = z_w.$$

We have $\mathbf{L}_{B|A} = J/J^2$ where J is the augmentation ideal of the simplicial B-algebra $X \otimes_A B$. Denote by N the normalization functor from simplicial B-modules. We have

$$(N(J/J^2))_3 = \bigoplus_w BZ_w, \qquad (N(J/J^2))_2 = \bigoplus_v BS_v$$

and the image of the differential d_3 of $N(J/J^2)$ coincides with the image of the canonical homomorphism [A$_2$, Remarque 23]

$$\pi_2(F) \to \bigoplus_v BS_v.$$

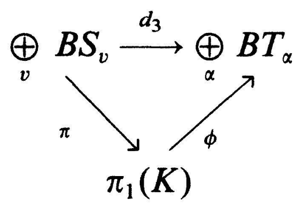
Therefore [A_2, Proposition 24]

$$\text{Coker}\, d_3 = \pi_1(K).$$

Moreover

$$(N(J/J^2))_1 = \bigoplus_\alpha BT_\alpha$$

and the differential d_3 places in a commutative diagram



where π is the homomorphism sending S_v on the generator represented by s_v and ϕ the canonical homomorphism [A$_2$, Remarque 23].

On the other hand

$$\pi_1(K) = \frac{M \cap N}{MN} = \text{Tor}_1^{K_1}(A, A)$$

where in the first variable in Tor is the structure given by ε_1^0 and in the second the one given by ε_1^1. If E denotes the Koszul complex associated to the elements t_α, then this Tor is isomorphic to $H_1(E)$. Moreover, through this isomorphism $\pi_1(K) = H_1(E)$, the homomorphism $\phi : \pi_1(K) \to \bigoplus_\alpha BT_\alpha$ corresponds to the canonical homomorphism

$$H_1(E) \longrightarrow E_1 \otimes_A B = \bigoplus_\alpha BT_\alpha$$

induced by the inclusions of cycles and boundaries $Z_1(E) \subset E_1$, $B_1(E) \subset IE_1$. Thus we have the following proposition:

PROPOSITION 8. *Let A be a ring, I an ideal of A, $B = A/I$ and E the Koszul complex associated to an arbitrary set of generators of I. Then we can choose $\mathbf{L}_{B|A}$ satisfying*:

(i) *The cokernel of the differential d_3 of $\mathbf{L}_{B|A}$ is a B-module isomorphic to $H_1(E)$.*

(ii) *There exists a morphism of complexes*

$$\mathbf{L}_{B|A} \longrightarrow (H_1(E) \xrightarrow{\phi} E_1 \otimes_A B)$$

where the second complex is concentrated in degrees 2 and 1. This morphism induces isomorphisms in homology in dimensions ≤ 2.

Now the cohomological part of Corollary 3 follows from the homological part and Proposition 8.

Remark 9. From Theorem 1 it follows that $H_j(A, B, C, -) = 0$ for all $j \geq 2$ if and only if $\mathrm{Tor}_p^A(B, C) = 0$ for all $p \geq 1$. This result is due to André [A_2, Remarque 39].

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