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## On nonpositively curved Euclidean submanifolds: splitting results

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**Abstract.** In this article, we prove that a  $n$ -dimensional, non-positively curved Euclidean submanifold with codimension  $p$  and with minimal index of relative nullity  $\nu = n - 2p$  is (in an open dense subset) locally the product of  $p$  hypersurfaces.

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Let  $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$  be an isometric immersion from a Riemannian manifold into a complete simply connected Riemannian manifold of constant sectional curvature  $c$  (superscripts will always denote dimensions). Denote by  $\nu$  the *index of relative nullity* of  $f$ ,

$$\nu(x) = \dim\{X \in T_x M : \alpha_f(X, Y) = 0, \forall Y \in T_x M\},$$

where  $\alpha_f$  stands for the vector valued second fundamental form of  $f$ . It is well known that having  $\nu > 0$  imposes strong restrictions on the manifold  $M^n$  and on its isometric immersion  $f$ . In [F1], the first author proved the inequality  $\nu \geq n - 2p$  when the sectional curvature of  $M^n$  satisfies  $K_M \leq c$  and gave several applications of this result. First let us show that this estimate is sharp.

**Example.** For each  $i = 1, \dots, p$ , let  $S_i \subseteq \mathbb{R}^3$  be a negatively curved surface. Then the product  $M^{2p} = S_1 \times \dots \times S_p \subseteq \mathbb{R}^{3p}$  satisfies the equality  $\nu = n - 2p = 0$ .

More generically, let  $M_i^{n_i} \subseteq \mathbb{R}^{n_i+1}$  be nowhere flat nonpositively curved hypersurfaces,  $i = 1, \dots, p$ . The Gauss equation tells us that the relative nullity  $\nu_i$  of  $M_i^{n_i}$  is  $\nu_i = n_i - 2$ . Then, the product manifold  $M^n = M_1^{n_1} \times \dots \times M_p^{n_p} \subseteq \mathbb{R}^{n+p}$  also have  $\nu = n - 2p$ .

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The first author proved in [F2] a general splitting theorem for Euclidean submanifolds  $f$  of nonpositive sectional curvature, under the additional assumption that the normal bundle of  $f$  is flat. The main purpose of this paper is to drop that assumption in the borderline case  $\nu = n - 2p$  to prove that the above example is essentially the unique one with minimal relative nullity index.

**Theorem 1.** *Let  $f : M^n \rightarrow \mathbb{R}^{n+p}$  be an isometric immersion into Euclidean space of a Riemannian manifold with nonpositive sectional curvature. Assume that  $\nu = n - 2p$  everywhere. Then there exists an open dense subset  $\mathcal{U} \subset M^n$  such that  $f|_{\mathcal{U}}$  splits locally as a product of  $p$  Euclidean hypersurfaces, that is, for any  $x \in \mathcal{U}$ , there exist a neighborhood  $x \in \mathcal{V} \subseteq \mathcal{U}$  and  $p$  nowhere flat Euclidean hypersurfaces  $f_i : M_i^{n_i} \rightarrow \mathbb{R}^{n_i+1}$  of nonpositive sectional curvature, such that*

$$\mathcal{V} = M_1 \times \cdots \times M_p \quad \text{and} \quad f|_{\mathcal{V}} = f_1 \times \cdots \times f_p$$

*split.*

First of all, note that when  $f$  is analytic, the splitting occurs on the entire  $M$ . In the general case, each  $n_i$  is constant in a connected components of  $\mathcal{U}$ , in fact, the universal covering space of any component of  $\mathcal{U}$  is the product of  $p$  Euclidean hypersurfaces. However, there are examples in which the  $n_i$ 's are not constant in the entire  $\mathcal{U}$ . Secondly, it is interesting to observe that, from Theorem 1 of [M] we have that  $f|_{\mathcal{V}}$  in the above is isometrically rigid if and only if each factor is rigid.

**Corollary 2.** *Let  $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$ ,  $2p \leq n$ , be an isometric immersion of a connected Riemannian manifold  $M^n$  with  $K_M \leq c$  and Ricci curvature  $\text{Ric}_M < c$ . Then  $c = 0$ ,  $n = 2p$  and  $f$  splits locally as a product of  $p$  negatively curved surfaces of  $\mathbb{R}^3$ . Moreover, the splitting is global provided that  $M^n$  is a Hadamard manifold.*

The assumption on the Ricci curvature in the above can be replaced by the weaker one  $\nu = 0$ . Also, the Hadamard condition can probably be relaxed a bit. Combining our results and [Z], we can state the complex analogue of the above:

**Theorem 3.** *Let  $X^n$  be an immersed complex submanifold of  $\mathbb{C}\mathbb{Q}_c^{n+p}$ , the complex space form of constant holomorphic sectional curvature  $c$ . Assume that  $X^n$  has nonpositive extrinsic sectional curvature. Then the index of relative nullity of  $X^n$  satisfies  $\nu \geq n - p$  and:*

(1) *when  $\nu = n - p = 0$ , we must have  $c = 0$ ;*

(2) *when  $c = 0$  and  $\nu = n - p$ ,  $X^n$  is locally holomorphically isometric to a product*

$$\mathbb{C}^k \times X^{n_1} \times \cdots \times X^{n_p} \subseteq \mathbf{X}^{n+p}, \quad n = k + \sum_{i=1}^p n_i,$$

*for some  $0 \leq k \leq \nu$ , where each  $X^{n_i} \subseteq \mathbb{C}^{n_i+1}$  is a nowhere flat nonpositively curved hypersurface.*

Moreover, if  $X^n$  is complete, then its universal covering is holomorphically isometric to the product  $\mathbb{C}^\nu \times \Sigma_1 \times \cdots \times \Sigma_p$ , where each  $\Sigma_i \hookrightarrow \mathbb{C}^2$  is a complete immersion of the unit disc. All dimensions here are the complex ones.

Notice that the real analyticity of  $X^n$  prevented  $k$  from jumping around. The last part of Theorem 3 is because, by a theorem of Abe in [A], any complete immersed complex submanifold of  $\mathbb{C}^m$  with one dimensional Gauss image must be a cylinder.

**Remark.** Any Euclidean hypersurface  $g : H^m \rightarrow \mathbb{R}^{m+1}$  of nonpositive sectional curvature without flat points can be described locally by means of the Gauss parametrization in the following way (see [DG] for details). Take a surface  $\xi : V^2 \rightarrow \mathbb{S}^m$  in the Euclidean unit sphere and a smooth function  $\gamma$  on  $V^2$ . The map  $\Psi : T_\xi^\perp V \rightarrow \mathbb{R}^{m+1}$  given by

$$\Psi(v) = \gamma\xi + \text{grad } \gamma + v$$

parametrizes  $g$  over the normal bundle of  $\xi$ , in the open set of normal vectors  $v$  which satisfies  $\det(\gamma\text{Id} + \text{Hess}_\gamma - B_v) < 0$ . Here,  $B_v$  denotes the second fundamental operator of  $\xi$  in the direction  $v$ . In this parametrization,  $\xi$  is the Gauss map of  $g$  and  $\gamma = \langle g, \xi \rangle$  its support function. For a discussion on the isometric deformations of those hypersurfaces see [DFT]. Observe that any isometric immersion  $f$  as in Theorem 1 can now be explicitly parametrized locally along  $\mathcal{U}$  using the Gauss parametrization for each factor.

### The flatness of the normal bundle

Let  $\alpha : V^n \times V^n \rightarrow W^p$  be a symmetric bilinear map, where  $V$  and  $W$  are real vector spaces of dimension  $n$  and  $p$ , respectively, and  $W$  is equipped with an inner product  $\langle \cdot, \cdot \rangle$ . Assume  $\alpha$  is *nonpositive* as defined in [F1], i.e.,

$$K_\alpha(X, Y) = \langle \alpha(X, X), \alpha(Y, Y) \rangle - \|\alpha(X, Y)\|^2 \leq 0,$$

for all  $X, Y \in V$ . Denote by  $\nu$  the dimension of the null space  $N$  of  $\alpha$ :

$$N = \{X \in V \mid \alpha(X, Y) = 0, \forall Y \in V\}.$$

Recall that a subspace  $T \subseteq V$  is said to be *asymptotic*, if  $\alpha(X, Y) = 0$  for all  $X, Y \in T$ . We know from [F1] that, for the above  $\alpha$ ,  $\nu \geq n - 2p$ . The main technical part of this article is the following diagonalization result for the borderline case  $\nu = n - 2p$ .

**Proposition 4.** *Let  $\alpha : V^n \times V^n \rightarrow W^p$  be a symmetric, nonpositive bilinear map. If  $\nu = n - 2p$ , then there exist a basis  $\{e_1, \dots, e_n\}$  of  $V$  and an orthonormal*

basis  $\{w_1, \dots, w_p\}$  of  $W$  such that  $\{e_{2p+1}, \dots, e_n\}$  is a basis of the null space  $N$ , and for each  $i, j \leq 2p$ ,

$$\alpha(e_i, e_j) = \delta_{ij}(-1)^i w_{\lfloor \frac{i+1}{2} \rfloor}.$$

*Proof.* We will carry out the induction on  $p$ . When  $p = 1$ ,  $\alpha$  is just a symmetric bilinear form, so it can always be diagonalized. The nonpositivity condition will force the rank of  $\alpha$  to be less or equal than 2, and when it equals 2, the two nonzero eigenvalues must be of opposite sign. Now assume that the result holds when  $\dim W < p$ , and consider the case  $\dim W = p$ .

By restricting  $\alpha$  to a subspace  $\tilde{V}^{2p}$  such that  $V = N \oplus \tilde{V}$ , we may assume that  $n = 2p$  and  $\nu = 0$ . Denote by  $\alpha_X$  the endomorphism  $\alpha_X(Y) = \alpha(X, Y)$ . By Proposition 6 of [F1] we know that there exists an asymptotic subspace  $T^p \subseteq \tilde{V}^{2p}$  of  $\alpha$ . Set

$$r = \min\{\text{rank } \alpha_X : 0 \neq X \in T\} > 0.$$

Fix a vector  $X \in T$  with  $\text{rank } \alpha_X = r$  and let  $V' = \text{Ker}(\alpha_X) \supseteq T$ . Thus, by the first claim in the proof of Proposition 6 of [F1], we know that the image  $\alpha(V' \times V')$  is perpendicular to the image subspace  $\text{Im}(\alpha_X)$ , that is, we have the restriction map

$$\alpha|_{V' \times V'}: V' \times V' \rightarrow \text{Im}(\alpha_X)^\perp.$$

Let  $N'$  be its null space. If there is  $Y \in N' \setminus T$ , then  $\text{span}(T \cup \{Y\})$  would be an asymptotic subspace of  $\alpha$  of dimension  $p + 1$ . By Proposition 8 of [F1], we get  $\nu \geq 1$ , a contradiction to our assumption. Therefore,  $N' \subseteq T$ .

For each  $Y \in N' \subseteq T$ , we have  $\text{Ker}(\alpha_Y) \supseteq V' = \text{Ker}(\alpha_X)$ , so  $\text{rank } \alpha_Y = r$ . Therefore,

$$V' = \text{Ker}(\alpha_Y), \quad \forall 0 \neq Y \in N'. \quad (1)$$

Put  $W_0 = \text{span}\{\text{Im}(\alpha_Y) : Y \in N'\}$  which has dimension  $r + s$ , for some  $s \geq 0$ . Again from the proof of Proposition 6 of [F1], we know that  $\alpha(V' \times V')$  is perpendicular to  $W_0$ , that is,

$$\beta = \alpha|_{V' \times V'}: V' \times V' \rightarrow W_0^\perp$$

is itself a symmetric, nonpositive bilinear map, with  $\dim V' = 2p - r$ ,  $\dim W_0^\perp = p - r - s$ . Write  $q = \dim N'$ . Then by Proposition 9 of [F1] we have

$$q \geq (2p - r) - 2(p - r - s) = r + 2s. \quad (2)$$

On the other hand, if  $\{Y_1, \dots, Y_q\}$  is a basis of  $N'$  and  $Z \in V \setminus V'$ , from (1) we obtain that the set of vectors  $\{\alpha(Y_1, Z), \dots, \alpha(Y_q, Z)\}$  in  $W_0$  must be linearly independent. Thus

$$q \leq r + s. \quad (3)$$

We conclude from (2) and (3) that  $s = 0$  and  $q = r$ . So we can apply the induction hypothesis on  $\beta$ . However, we want to show first that  $r = 1$ .

Assume the contrary, that is,  $q > 1$ . Take a subspace  $V_1^r$  such that  $V_1 \oplus V^r = V$ . Choose any  $Y \in N^r$  not collinear with  $X$ . Since  $s = 0$ , (the restriction of) both  $\alpha_X$  and  $\alpha_Y$  give isomorphisms between  $V_1$  and  $W_0^\perp$ . Fix an orthonormal basis  $\{w_1, \dots, w_r\}$  of  $W_0^\perp$ . Let  $\{v_1, \dots, v_r\}$  be the basis of  $V_1$  such that  $\alpha_X(v_i) = w_i$  and write  $\alpha_Y(v_i) = \sum_{j=1}^r B_{ij}w_j$ . That is, we identify  $V_1$  and  $W_0^\perp$  through  $\alpha_X$ , and use the matrix  $B$  to represent  $\alpha_Y$ .

If  $B$  has a real eigenvalue  $\lambda$ , then  $\alpha_{Y-\lambda X}$  would have rank less than  $r$ , which contradicts (1). So the matrix  $B$  has no real eigenvalues. By considering a complex eigenvector which corresponds to a complex eigenvalue of  $B$ , we obtain two 2-planes  $P \subseteq V_1, Q \subseteq W_0^\perp$ , such that both  $\alpha_X$  and  $\alpha_Y$  give isomorphisms between  $P$  and  $Q$ .

Now let us fix an orthonormal basis  $\{w_1, w_2\}$  of  $Q$ , and let  $\{e_3, e_4\}$  be the basis of  $P$  such that  $\alpha_X(e_3) = w_1, \alpha_X(e_4) = w_2$ . Write

$$\alpha_Y(e_3) = aw_1 + bw_2, \quad \alpha_Y(e_4) = cw_1 + dw_2.$$

Replacing  $Y$  by  $Y - dX$ , we may assume that

$$d = 0.$$

We know that the  $2 \times 2$  real matrix with entries  $a, b, c, 0$  can not have any real eigenvalue, or equivalently,

$$4bc + a^2 < 0.$$

Set  $e_1 = X, e_2 = Y$ . For arbitrary real constants  $x$  and  $y$ , let us consider the vectors  $Z = xe_1 + ye_2 + xe_3 - e_4$  and  $Z' = ye_2 + e_3$ . We have

$$Z \wedge Z' = xye_1 \wedge e_2 + xe_1 \wedge e_3 + ye_2 \wedge e_4 + e_3 \wedge e_4.$$

Define the symmetric bilinear form  $R$  on  $\Lambda^2V$ , the curvature of  $\alpha$ , as

$$R(Z_1 \wedge Z_2, Z_3 \wedge Z_4) = \langle \alpha(Z_1, Z_3), \alpha(Z_2, Z_4) \rangle - \langle \alpha(Z_1, Z_4), \alpha(Z_2, Z_3) \rangle. \quad (4)$$

Hence, the matrix of  $R$  under the partial basis  $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$  is

$$R = \begin{bmatrix} 0 & 0 & 0 & c-b \\ 0 & -1 & -b & -f \\ 0 & -b & -c^2 & -g \\ c-b & -f & -g & -h \end{bmatrix}.$$

Therefore  $-R(Z \wedge Z', Z \wedge Z') = x^2 + c^2y^2 + h + 2(2b - c)xy + 2fx + 2gy$ . Thus, the nonpositivity of  $\alpha$  gives us

$$c^2y^2 + 2((2b - c)x + g)y + (x^2 + 2fx + h) \geq 0.$$

Hence, the discriminant with respect to  $y$  must be nonpositive, that is,

$$0 \leq c^2(x^2 + 2fx + h) - ((2b - c)x + g)^2 = (4bc - 4b^2)x^2 + 2(c^2f + cg - 2bg)x + (c^2h - g^2).$$

Since  $a^2 + 4bc < 0$ , the leading coefficient is negative, which is a contradiction for  $x$  sufficiently large. This completes the proof of the claim that  $q = r = 1$ .

Now applying the induction hypothesis on the restriction map  $\beta$ , we obtain an orthonormal basis  $\{w_1, \dots, w_p\}$  of  $W$  and a basis  $\{e'_1, e_2, e'_2, \dots, e_p, e'_p\}$  of  $V' = \text{Ker}(\alpha_X)$  such that  $X = e'_1$ ,  $\text{Im}(\alpha_X) = \text{span}\{w_1\}$ ,

$$\alpha(e_i, e_j) = \delta_{ij}w_i, \quad \alpha(e'_i, e'_j) = -\delta_{ij}w_i, \quad \alpha(e_i, e'_j) = 0, \quad \forall 2 \leq i, j \leq p,$$

and of course  $\alpha(e'_1, e'_1) = \alpha(e'_1, e_i) = \alpha(e'_1, e'_i) = 0$ , for all  $2 \leq i \leq p$ .

Choose a vector  $e_1 \in V \setminus V'$  such that  $\alpha(e_1, e'_1) = w_1$ . Write  $\alpha = (A^1, \dots, A^p)$ , where each  $A^k_{ab} = \langle \alpha(e_a, e_b), w_k \rangle$  is a symmetric  $2p \times 2p$  matrix. Here for convenience we adopt the notations  $e'_i = e_{p+i}$  and  $i' = i + p$ , for  $i \leq p$ . Under the basis  $\{e_a \wedge e_b; 1 \leq a < b \leq 2p\}$  of  $\Lambda^2 V$ , the coordinate matrix of the bilinear form  $R$  becomes

$$R_{ab,cd} = \sum_{k=1}^p (A^k_{ac}A^k_{bd} - A^k_{ad}A^k_{bc}).$$

The nonpositivity of  $\alpha$  simply says that  $R(Z_1 \wedge Z_2, Z_1 \wedge Z_2) \leq 0$ . For any three vectors  $Z_i$ ,  $i = 1, 2, 3$ , by considering the nonpositivity at  $Z_1 \wedge (Z_2 + xZ_3)$  for arbitrary  $x$ , we have

$$R(Z_1 \wedge Z_2, Z_1 \wedge Z_2) \cdot R(Z_1 \wedge Z_3, Z_1 \wedge Z_3) \geq (R(Z_1 \wedge Z_2, Z_1 \wedge Z_3))^2. \quad (5)$$

For all  $2 \leq i \leq p$  and  $2 \leq a \neq i, i'$ , from the above and  $R_{ia,ia} = 0$  we have  $R_{1i,ia} = -A^i_{1a} = 0$ . That is,  $A^i_{1j} = A^i_{1j'} = 0$ , for all  $2 \leq i \neq j \leq p$ . Replacing  $e_1$  by  $e_1 - \sum_{i=2}^p (A^i_{1i}e_i - A^i_{1i'}e'_i)$ , we may assume that

$$A^i_{1j} \equiv 0, \quad \forall i, j \geq 2. \quad (6)$$

For  $2 \leq i \leq p$ , set

$$b_i = A^i_{11}, \quad a_i = A^1_{1i}, \quad c_i = A^1_{1i'}.$$

Thus,

$$\begin{aligned} R_{11',11'} &= -1, \\ R_{1i,1i} &= b_i - a_i^2, \quad R_{11',1i} = -a_i, \\ R_{1i',1i'} &= b_i - c_i^2, \quad R_{11',1i'} = -c_i, \end{aligned}$$

since  $A^1_{11'} = 1$ . From (5) and  $R_{11',11'}R_{1i,1i} \geq (R_{11',1i})^2$  we get  $b_i \leq 0$ . Similarly, replacing  $i$  by  $i'$ , we have  $b_i \geq 0$ . Therefore, all  $b_i = 0$ .

Now we take any nonsingular  $2 \times 2$  matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} A_{11}^1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and set

$$\tilde{e}_1 = ae_1 + ce'_1, \quad \tilde{e}'_1 = be_1 + de'_1, \quad \tilde{e}_i = e_i - a_i e'_1, \quad \tilde{e}'_i = e'_i - c_i e_1, \quad 2 \leq i \leq p.$$

Then under the new basis  $\{\tilde{e}_a\}$  of  $V$ , we have  $\alpha(\tilde{e}_a, \tilde{e}_b) = 0$ , if  $a \neq b, b'$ , and

$$\alpha(\tilde{e}_i, \tilde{e}_i) = w_i, \quad \alpha(\tilde{e}'_i, \tilde{e}'_i) = -w_i, \quad \forall 1 \leq i \leq p.$$

This completes the proof of Proposition 4. □

Let us examine the diagonalizing frame  $\{w_i\}$  of Proposition 4. Set

$$\mathcal{D} = \{X \in V : \text{rank}(\alpha_X) \leq 1\}.$$

This set of course depends only on  $\alpha$ . By Proposition 4, we know that  $\mathcal{D}$  is the union of  $p$  subspaces of dimension  $\nu + 2$ , denoted by  $\mathcal{D}_i, i = 1, \dots, p$ , with  $\mathcal{D}_i \cap \mathcal{D}_j = N$  for all  $i \neq j$ . If we choose a plane  $V_i \subseteq \mathcal{D}_i$  which has trivial intersection with  $N$ , then  $V$  is the direct sum

$$V = N \oplus V_1 \oplus \dots \oplus V_p$$

and  $\alpha(\mathcal{D}_i \times \mathcal{D}_j) = 0$  if  $i \neq j$ , while all  $\alpha(\mathcal{D}_i \times \mathcal{D}_i)$  are one dimensional and mutually perpendicular. So the orthonormal frame  $\{w_i\}$  is uniquely determined up to permutations.

It is interesting to note that  $K \leq 0$  does not implies in general that the symmetric curvature operator  $R$  is negative semidefinite. However, it is easy to see using Proposition 4 that, in our case, we really have  $R \leq 0$ . In fact,  $\{e_i \wedge e_{i+p} : 1 \leq i \leq p\}$  is a basis of the orthogonal complement  $F$  of the nullity space of  $R$  in  $\Lambda^2 V$  formed by the unique (up to scaling) decomposable elements in  $F$ . Indeed,  $e_i \wedge e_{i+p}$  is eigenvector of  $R$  of eigenvalue  $K(e_i, e_{i+p}) \neq 0$ .

We are now in position to give the remaining proofs.

*Proofs of Theorem 1 and Corollary 2.* For each  $x \in M^n$ , consider  $\alpha_f(x)$  the vector valued second fundamental form of  $f$  at  $x$ . Since  $K_M \leq 0$ , the Gauss equation tells us that  $\alpha_f(x)$  is nonpositive. Thus, we apply Proposition 4 to it to obtain the



special (smooth) orthonormal frame  $\{w_i, 1 \leq i \leq p\}$ . By Theorem 1 and Corollary 2 of [F2], we only need to prove that the normal bundle of  $f$  is flat. We will show indeed that this frame is normal parallel.

For each  $1 \leq i \leq p$ , consider the shape tensor  $A_{w_i}$  on  $M^n$  defined by  $\langle A_{w_i}X, Y \rangle = \langle \alpha_f(X, Y), w_i \rangle$ . By Proposition 4,  $V_i = \text{Im } A_{w_i}$  are two dimensional distributions on  $M^n$  such that

$$V_1 \oplus \cdots \oplus V_p = \Delta^\perp, \quad (7)$$

where  $\Delta$  stands for the relative nullity distribution of  $f$ . Let  $\psi_{ij}$  be the 1-forms defined by  $\psi_{ij}(X) = \langle \nabla_X^\perp w_i, w_j \rangle$ . We only need to show that  $\psi_{ij} = 0$ , for all  $i, j$ .

Recall that the Codazzi equation for  $A_{w_i}$  is

$$\nabla_X(A_{w_i}Y) - A_{w_i}\nabla_XY - A_{\nabla_X^\perp w_i}Y = \nabla_Y(A_{w_i}X) - A_{w_i}\nabla_YX - A_{\nabla_Y^\perp w_i}X. \quad (8)$$

Taking in (8)  $X, Y \in V_i^\perp = \text{Ker } A_{w_i}$  we easily obtain using (7) that

$$A_{w_j}(\psi_{ij}(X)Y - \psi_{ij}(Y)X) = 0, \quad \forall X, Y \in V_i^\perp, \quad 1 \leq j \leq p.$$

Suppose that there is  $X_0 \in V_i^\perp$ , and  $j \neq i$  such that  $\psi_{ij}(X_0) \neq 0$ . The above equation implies that  $V_i^\perp \subset V_j^\perp \oplus \text{span } \{X_0\}$ , that is,

$$T_xM \neq V_i^\perp + V_j^\perp = (V_i \cap V_j)^\perp,$$

which is a contradiction by (7). Thus  $V_i^\perp \subset \text{Ker } \psi_{ij}$ , for all  $i, j$ . By the orthonormality of  $\{w_i\}$  we have  $\psi_{ij} = -\psi_{ji}$ . Therefore,  $T_xM = V_i^\perp + V_j^\perp \subset \text{Ker } \psi_{ij}$ . Notice that the Ricci equations imply that the  $V_i$ 's are orthogonal. This concludes our proof.  $\square$

The proof of Theorem 3 can be obtained by combining the diagonalization theorem of [Z] (together with the similar argument of the orthogonality of the special frame) and the proof of the Theorem 1 of [F2]. So we shall omit it here.

## Final comments

*i)* Let us explain Theorem 1 a little bit. We have everywhere on  $M^n$  the orthogonal decomposition  $TM = N \oplus V_1 \oplus \cdots \oplus V_p$  of the tangent bundle into distributions. Let  $\tilde{V}_i$  be the distribution spanned by all vector fields in  $V_i$  and all  $\nabla_{X_1} \cdots \nabla_{X_s} X_{s+1}$ , where all  $X_j \in V_i$ . It is shown in [F2] that  $\tilde{V}_i \perp \tilde{V}_j$  whenever  $i \neq j$ , and all  $\tilde{V}_i$  are parallel distributions (in the neighborhood where they have constant dimensions). Let  $n_i(x)$  be the dimension of  $\tilde{V}_i$  at  $x$ . Each  $n_i$  is a lower semicontinuous integer-valued function. If  $k = n - \sum_{i=1}^p n_i$ , then  $0 \leq k \leq \nu$ . Let  $\mathcal{U}$  be the open dense subset of  $M^n$  which is the disjoint union of open subsets  $\mathcal{U}_j$  in which  $k(x)$  takes constant value  $j$ . All  $n_i$  are necessarily constant in  $\mathcal{U}_j$ , and we have the desired

local splitting on  $U_j$ . Observe that, using the Gauss parametrization, it is easy to construct examples of submanifolds with the functions  $n_i$  nonconstant. Therefore, for  $\nu > 0$  we can only obtain the local splitting along an open dense subset. With this in mind, the same argument as in Corollary 2 of [F2] proves the following

**Theorem 5.** *Let  $f : M^n \rightarrow \mathbb{Q}_c^{2n-r}$ ,  $2 \leq r \leq n/2$ , be an isometric immersion with flat normal bundle of a connected Riemannian manifold with  $K_M \leq c$  and  $Ric_M < c$ . Then  $c = 0$  and  $f$  splits locally as a product of  $r$  nonpositively curved Euclidean submanifolds, that is,  $f = f_1 \times \dots \times f_r$  locally, with  $f_i : M_i^{n_i} \rightarrow \mathbb{R}^{2n_i-1}$ . The splitting is global provided  $M^n$  is a Hadamard manifold.*

Again, the assumption on the Ricci curvature can be replaced by  $\nu = 0$ .

ii) We believe that the case  $\nu = n - 2p > 0$  for an isometric immersion  $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$ , with  $c \neq 0$ , cannot occur. It would be interesting either to prove its nonexistence or to construct such an example. The complex case should be similar.

iii) Taking the curvature tensor  $R$  as a 4-tensor on  $M^n$ , it is defined the nullity space of  $M^n$  at  $x$  as the subspace  $\Gamma(x) = \{X \in T_x M : R(X, Y, Z, W) = 0, \forall Y, Z, W \in T_x M\}$ . This is an intrinsic subspace, so its dimension  $\mu(x)$  called the nullity index of  $M^n$  is an intrinsic function. For an isometric immersion  $f$  of  $M^n$  into Euclidean space we always have that the relative nullity distribution  $\Delta$  of  $f$  satisfies  $\Delta \subset \Gamma$ . Thus, our assumption on the relative nullity distribution in Theorem 1 can be replaced by the intrinsic one  $\mu = n - 2p$ . The same holds for Corollary 2.

iv) Now let us consider the more general situation discussed in Theorem 1 of [F2], namely,  $\nu = n - p - r$ , for some  $2 \leq r \leq p$ . It is natural to ask if it can be generalized by dropping the flatness of the normal bundle assumption as we did for the case  $r = p$ . The answer to this question seems to be negative, since the algebraic decomposition Proposition 4 does not generalize, even for the case  $r = p - 1$ , as the following example shows. Take  $A_i$  defined as

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The bilinear form  $\alpha = (A_1, A_2, A_3) : \mathbb{R}^5 \times \mathbb{R}^5 \rightarrow \mathbb{R}^3$  is nonpositive, has  $\nu = n - p - r = 0$  for  $r = p - 1 = 2$  but is not decomposable. It is easy to generalize this example for all  $p$ . Thus the analogous result to Proposition 4 is false for  $\nu = n - p - r$  and  $2 \leq r \leq p - 1$ .

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