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On the zero set of semi-invariants for tame quivers

Christine Riedtmann and Grzegorz Zwara

Abstract. Let \mathbf{d} be a prehomogeneous dimension vector for a finite tame quiver Q . We show that the common zeros of all non-constant semi-invariants for the variety of representations of Q with dimension vector $N \cdot \mathbf{d}$, under the product of the general linear groups at all vertices, is a complete intersection for $N \geq 3$.

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1. Introduction

Let k be an algebraically closed field, and let $Q = (Q_0, Q_1, t, h)$ be a finite quiver, i.e. a finite set $Q_0 = \{1, \dots, n\}$ of vertices and a finite set Q_1 of arrows $\alpha : t\alpha \rightarrow h\alpha$, where $t\alpha$ and $h\alpha$ denote the tail and the head of α , respectively.

A representation of Q over k is a collection $(X(i); i \in Q_0)$ of finite dimensional k -vector spaces together with a collection $(X(\alpha) : X(t\alpha) \rightarrow X(h\alpha); \alpha \in Q_1)$ of k -linear maps. A morphism $f : X \rightarrow Y$ between two representations is a collection $(f(i) : X(i) \rightarrow Y(i))$ of k -linear maps such that

$$f(h\alpha) \circ X(\alpha) = Y(\alpha) \circ f(t\alpha) \quad \text{for all } \alpha \in Q_1.$$

By $\sigma(X)$ we denote the number of pairwise non-isomorphic indecomposable direct summands occurring in a decomposition of X into indecomposables. According to the theorem of Krull–Schmidt, $\sigma(X)$ is well-defined. The dimension vector of a representation X of Q is the vector

$$\mathbf{dim} X = (\dim X(1), \dots, \dim X(n)) \in \mathbb{N}^{Q_0}.$$

We denote the category of representations of Q by $\text{rep}(Q)$, and for any vector $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^{Q_0}$

$$\text{rep}(Q, \mathbf{d}) = \prod_{\alpha \in Q_1} \text{Mat}(d_{h\alpha} \times d_{t\alpha}, k)$$

is the vector space of representations X of Q with $X(i) = k^{d_i}$, $i \in Q_0$. The group

$$\mathrm{Gl}(\mathbf{d}) = \prod_{i=1}^n \mathrm{Gl}(d_i, k)$$

acts on $\mathrm{rep}(Q, \mathbf{d})$ by

$$((g_1, \dots, g_n) \star X)(\alpha) = g_{h\alpha} \cdot X(\alpha) \cdot g_{t\alpha}^{-1}.$$

Note that the $\mathrm{Gl}(\mathbf{d})$ -orbit of X consists of the representations Y in $\mathrm{rep}(Q, \mathbf{d})$ which are isomorphic to X .

We call \mathbf{d} a prehomogeneous dimension vector if $\mathrm{Gl}(\mathbf{d}) \star T$ is an open orbit for some T in $\mathrm{rep}(Q, \mathbf{d})$. Such a representation T is characterized by $\mathrm{Ext}_Q^1(T, T) = 0$ [9]. If Q admits only finitely many indecomposable representations, or equivalently if the underlying graph of Q is a disjoint union of Dynkin diagrams of type \mathbb{A} , \mathbb{D} or \mathbb{E} [6], every vector \mathbf{d} is prehomogeneous. Indeed, any representation is a direct sum of indecomposables and therefore $\mathrm{rep}(Q, \mathbf{d})$ contains finitely many orbits, one of which must be open.

Let \mathbf{d} be prehomogeneous, and let $f_1, \dots, f_s \in k[\mathrm{rep}(Q, \mathbf{d})]$ be the irreducible monic polynomials whose zeros $Z(f_1), \dots, Z(f_s)$ are the irreducible components of codimension 1 of $\mathrm{rep}(Q, \mathbf{d}) \setminus \mathrm{Gl}(\mathbf{d}) \star T$, where $\mathrm{Gl}(\mathbf{d}) \star T$ is the open orbit. It is easy to see that

$$g \cdot f_i = \chi_i(g) \cdot f_i$$

for $g \in \mathrm{Gl}(\mathbf{d})$, where $\chi_i : \mathrm{Gl}(\mathbf{d}) \rightarrow k^*$ is a character. A regular function with this property is called a semi-invariant. By [11], any semi-invariant is a scalar multiple of a monomial in f_1, \dots, f_s , and f_1, \dots, f_s are algebraically independent. We denote by

$$\mathcal{Z}_{Q, \mathbf{d}} = \{X \in \mathrm{rep}(Q, \mathbf{d}); f_i(X) = 0, i = 1, \dots, s\}$$

the closed subscheme of $\mathrm{rep}(Q, \mathbf{d})$ of common zeros of all non-constant semi-invariants. Obviously we have $\mathrm{codim} \mathcal{Z}_{Q, \mathbf{d}} \leq s$, and equality means that $\mathcal{Z}_{Q, \mathbf{d}}$ is a complete intersection.

Let T_1, \dots, T_r be pairwise non-isomorphic indecomposable representations of Q such that $\mathrm{Ext}_Q^1(T_i, T_j) = 0$ for any $i, j \leq r$. In [8] we showed that there is a positive integer N such that $\mathcal{Z}_{Q, \mathbf{d}}$ is a complete intersection and irreducible for any dimension vector $\mathbf{d} = \sum_{i=1}^r \lambda_i \mathbf{dim} T_i$ with $\lambda_i \geq N$, $i = 1, 2, \dots, r$. Now our goal is to prove that N is quite small in case Q is tame; i.e., every connected component Δ of Q is either a Dynkin quiver or an extended Dynkin quiver. Our methods are completely different.

Assume that Q is tame, and set

$$N(Q) = \max N(\Delta),$$

where Δ ranges over the connected components of Q and where

$$N(\Delta) = \begin{cases} 1 & \text{if } |\Delta| = \mathbb{A}_m \text{ or } \tilde{\mathbb{A}}_m, \\ 2 & \text{if } |\Delta| = \mathbb{D}_m, \mathbb{E}_6, \mathbb{E}_7 \text{ or } \mathbb{E}_8, \\ 3 & \text{if } |\Delta| = \tilde{\mathbb{D}}_m, \tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7 \text{ or } \tilde{\mathbb{E}}_8, \end{cases}$$

and $|\Delta|$ denotes the underlying non-oriented graph of the quiver Δ . Note that $N(K) \leq N(Q)$ for any subquiver K of Q .

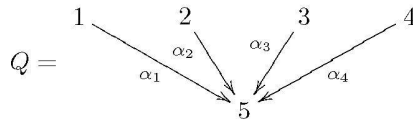
Theorem 1.1. *Suppose Q is tame. Let T_1, \dots, T_r be pairwise non-isomorphic indecomposable representations of Q such that $\text{Ext}_Q^1(T_i, T_j) = 0$ for any $i, j \leq r$. Choose positive integers $\lambda_1, \dots, \lambda_r$ and set $\lambda = \min \lambda_i$, $\mathbf{d} = \sum_{i=1}^r \lambda_i \mathbf{dim} T_i$. Then $\mathcal{Z}_{Q, \mathbf{d}}$ is*

- (i) *a complete intersection provided $\lambda \geq N(Q)$,*
- (ii) *irreducible provided $\lambda \geq N(Q) + 1$.*

Note that the case of a Dynkin quiver of type \mathbb{A}_n has been treated by Chang and Weyman in [5].

In case k is the field \mathbb{C} of complex numbers, the fact that $\mathcal{Z}_{Q, \mathbf{d}}$ is a complete intersection implies that $\text{rep}(Q, \mathbf{d})$ is cofree as a representation of the subgroup $\text{Sl}(\mathbf{d}) = \prod_{i=1}^n \text{Sl}(d_i)$ of $\text{Gl}(\mathbf{d})$; i.e., $\mathbb{C}[\text{rep}(Q, \mathbf{d})]$ is a free module over the ring $\mathbb{C}[\text{rep}(Q, \mathbf{d})]^{\text{Sl}(\mathbf{d})}$ of $\text{Sl}(\mathbf{d})$ -invariant polynomials [13, §17], [8].

Example. Let us consider the quiver



with the dimension vector $\mathbf{d} = \lambda \cdot \mathbf{e}$, $\lambda \in \mathbb{N}$ and $\mathbf{e} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ & & & 3 \end{pmatrix}$ as an example.

There is an indecomposable representation T_1 in $\text{rep}(Q, \mathbf{e})$, whose orbit is open. The complement of the open orbit of $T = T_1^\lambda$ in $\text{rep}(Q, \mathbf{d})$ has 4 components of codimension 1, defined by

$$\det \left(X(\alpha_1) \cdots \widehat{X(\alpha_j)} \cdots X(\alpha_4) \right) = 0,$$

$j = 1, 2, 3, 4$, where the hat means “omit $X(\alpha_j)$ ”. Using the results developed later, we know that X belongs to $\mathcal{Z}_{Q, \mathbf{d}}$ if and only if X either contains the simple projective P_5 or else the direct sum $\bigoplus_{j=1}^4 P_j$ of the two-dimensional projectives associated to the vertices $1, \dots, 4$ as a direct summand. It is easy to check that

- $\mathcal{Z}_{Q, \mathbf{e}}$ is irreducible of codimension 2,
- $\mathcal{Z}_{Q, 2\mathbf{e}}$ has two irreducible components of codimension 3 and 4, respectively,
- $\mathcal{Z}_{Q, 3\mathbf{e}}$ has two irreducible components of codimension 4,
- $\mathcal{Z}_{Q, \lambda \cdot \mathbf{e}}$ is irreducible of codimension 4 for $\lambda \geq 4$.

2. Notations and preliminaries

The varieties considered in this paper are locally closed subsets of a k -vector space. If $\mathcal{A} \subseteq \mathcal{B}$ are two such varieties and \mathcal{B} is irreducible, we denote by $\text{codim}_{\mathcal{B}} \mathcal{A}$ the codimension of \mathcal{A} in \mathcal{B} . In case $\mathcal{B} = \text{rep}(Q, \mathbf{d})$, we omit the subscript \mathcal{B} .

We will assume throughout that the representation $T = \bigoplus_{i=1}^r T_i^{\lambda_i}$ is sincere, i.e., $T(l) \neq 0$ for any $l \in Q_0$. As the full subquiver K of Q which supports T is still tame with $N(K) \leq N(Q)$, this is no restriction. The assumption excludes oriented cycles as subquivers of Q . Indeed, a sincere representation of an oriented cycle cannot have an open orbit.

The Euler form of Q is the \mathbb{Z} -bilinear form on \mathbb{Z}^{Q_0} defined by

$$\langle \mathbf{d}, \mathbf{e} \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha \in Q_1} d_{t\alpha} e_{h\alpha}.$$

For $X \in \text{rep}(Q, \mathbf{d})$, $Y \in \text{rep}(Q, \mathbf{e})$ it can be computed as

$$\langle \mathbf{d}, \mathbf{e} \rangle = [X, Y] - {}^1[X, Y],$$

where

$$[X, Y] = \dim_k \text{Hom}_Q(X, Y) \quad \text{and} \quad {}^1[X, Y] = \dim_k \text{Ext}_Q^1(X, Y).$$

The quadratic form

$$q(\mathbf{d}) = \langle \mathbf{d}, \mathbf{d} \rangle$$

associated with the Euler form is the Tits form of Q . It is positive semi-definite as Q is tame and positive definite if Q does not contain extended Dynkin diagrams.

We follow Schofield [12] in order to describe the semi-invariants of $\text{rep}(Q, \mathbf{d})$: For a representation U of Q , the right perpendicular category U^\perp is the full subcategory of $\text{rep}(Q)$ whose objects are

$$\{Y; [U, Y] = {}^1[U, Y] = 0\}.$$

Dually, ${}^\perp U$ has as objects

$$\{Z; [Z, U] = {}^1[Z, U] = 0\}.$$

Note that $U^\perp = {}^\perp(\tau U)$, where τ is the Auslander–Reiten translation for all non-projective indecomposable direct summands of U and $\tau(P_l) = I_l$, where P_l and I_l are the projective and injective indecomposable representations associated to the vertex $l \in Q_0$, respectively. If ${}^1[U, U] = 0$, the category U^\perp is equivalent to the category of representations of a quiver with $n - \sigma(U)$ vertices.

Thus U^\perp contains $n - r$ simple objects if $T = \bigoplus_{i=1}^r T_i^{\lambda_i}$ is a representation of Q as in the statement of the theorem. If S is one of them, the set

$$\{X \in \text{rep}(Q, \mathbf{d}); [X, S] \neq 0\}$$

is a component of codimension 1 of the complement

$$\text{rep}(Q, \mathbf{d}) \setminus \text{Gl}(\mathbf{d}) \star T.$$

Non-isomorphic simple objects lead to distinct components, and all components of codimension 1 are obtained in this way. Thus $\mathcal{Z}_{Q,\mathbf{d}}$ is the zero set of $n - r$ (algebraically independent) polynomials. From now on, we will denote the underlying reduced variety of $\mathcal{Z}_{Q,\mathbf{d}}$ by the same symbol. This will cause no confusion since we are only interested in the irreducibility and the dimension of $\mathcal{Z}_{Q,\mathbf{d}}$. We have the following descriptions:

$$\begin{aligned} \mathcal{Z}_{Q,\mathbf{d}} &= \{X \in \text{rep}(Q, \mathbf{d}); [X, S] \neq 0 \text{ for all simple objects } S \in T^\perp\} \\ &= \{X \in \text{rep}(Q, \mathbf{d}); [S', X] \neq 0 \text{ for all simple objects } S' \in {}^\perp T\}. \end{aligned}$$

The material presented here can be found in [12]; compare also [8]. In order to obtain part (i) of our theorem it suffices to prove $\text{codim } \mathcal{Z}_{Q,\mathbf{d}} \geq n - r$.

Fix a sink $z \in Q_0$; i.e., a vertex z which is the head of some arrows $\alpha_j : y_j \rightarrow z$, $j = 1, \dots, s$, but the tail of none. The vertices y_1, \dots, y_s need not be distinct. Let E be the simple projective supported at z . By \overline{Q} we denote the full subquiver of Q with $\overline{Q}_0 = Q_0 \setminus \{z\}$ and by $\overline{\mathbf{d}}$ the restriction of \mathbf{d} to \overline{Q}_0 . Note that the orbit of the restriction $\overline{T} = \bigoplus_{i=1}^r \overline{T}_i^{\lambda_i}$ to \overline{Q} is open in $\text{rep}(\overline{Q}, \overline{\mathbf{d}})$. As E is the simple projective supported at z , we have

$$E^\perp = \{X \in \text{rep}(Q); X(z) = 0\},$$

which we identify with $\text{rep}(\overline{Q})$. There is a short exact sequence

$$0 \rightarrow E^{d_z} \rightarrow T \rightarrow \overline{T} \rightarrow 0.$$

Considering the long exact sequence of Hom's and Ext¹'s from it, we find that $E^\perp \cap T^\perp = E^\perp \cap \overline{T}^\perp = \overline{T}^\perp$.

We decompose $\mathcal{Z}_{Q,\mathbf{d}}$ as a disjoint union

$$\mathcal{Z}_{Q,\mathbf{d}} = \mathcal{Z}'_{Q,\mathbf{d}} \cup \mathcal{Z}''_{Q,\mathbf{d}},$$

where

$$\mathcal{Z}'_{Q,\mathbf{d}} = \{X \in \mathcal{Z}_{Q,\mathbf{d}}; [X, E] = 0\} \quad \text{and} \quad \mathcal{Z}''_{Q,\mathbf{d}} = \{X \in \mathcal{Z}_{Q,\mathbf{d}}; [X, E] \neq 0\}.$$

We will estimate the codimensions of $\mathcal{Z}'_{Q,\mathbf{d}}$ and $\mathcal{Z}''_{Q,\mathbf{d}}$ in $\text{rep}(Q, \mathbf{d})$ separately.

Throughout the article, $T = \bigoplus_{i=1}^r T_i^{\lambda_i}$ will denote a sincere representation of a tame quiver Q , and we set $\lambda = \min \lambda_i \geq 1$ and $\mathbf{dim} T = \mathbf{d}$.

3. The variety $\mathcal{Z}''_{Q,\mathbf{d}}$

Proposition 3.1. *A representation X in $\mathcal{Z}_{Q,\mathbf{d}}$ belongs to $\mathcal{Z}''_{Q,\mathbf{d}}$ if and only if*

(i) *the restriction \overline{X} to \overline{Q} lies in $\mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}}$*

and

(ii) $\text{rank}(X(\alpha_1) \cdots X(\alpha_s)) < d_z$.

In particular,

$$\text{codim } \mathcal{Z}''_{Q,\mathbf{d}} = \text{codim}_{\text{rep}(\overline{Q},\overline{\mathbf{d}})} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} + \max\left(0, \left(\sum_{j=1}^s d_{y_j}\right) - d_z + 1\right).$$

Proof. The second condition just says that E is a direct summand of X , or equivalently that $[X, E] \neq 0$. A representation $X = X' \oplus E$ belongs to $\mathcal{Z}_{Q,\mathbf{d}}$ if and only if

$$[X, S] = [X', S] + [E, S] > 0$$

for any simple object $S \in T^\perp$. Equivalently,

$$[X', S] > 0$$

holds for any simple representation $S \in T^\perp$ with $[E, S] = \dim S(z) = 0$. These are precisely the simple objects of $\overline{T}^{\perp\overline{Q}}$, and moreover we have

$$[X', S] = [\overline{X}', S] = [\overline{X}, S] > 0$$

since $S(z) = 0$.

As for the statement about $\text{codim } \mathcal{Z}''_{Q,\mathbf{d}}$, observe that, in case $d_z > \sum_{j=1}^s d_{y_j}$, any $d_z \times \sum_{j=1}^s d_{y_j}$ -matrix has rank less than d_z , whereas for $d_z \leq \sum_{j=1}^s d_{y_j}$, the subvariety

$$\mathcal{N}_{\mathbf{d}} = \left\{ A \in \text{Mat} \left(d_z \times \sum_{j=1}^s d_{y_j} \right); \text{rank } A < d_z \right\}$$

is of codimension $\left(\sum_{j=1}^s d_{y_j}\right) - d_z + 1$. □

Corollary 3.2. *Suppose that $\lambda \geq N(Q)$ and that E is not a direct summand of T .*

(i) *We have*

$$\text{codim } \mathcal{Z}''_{Q,\mathbf{d}} - n + \sigma(T) \geq \text{codim } \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} - (n - 1) + \sigma(\overline{T}).$$

(ii) *If moreover $d_z < \sum_{j=1}^s d_{y_j}$, we have*

$$\text{codim } \mathcal{Z}''_{Q,\mathbf{d}} - n + \sigma(T) \geq \text{codim } \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} - (n - 1) + \sigma(\overline{T}) + \lambda - N(Q).$$

In order to prove this result, we need some information about the number $\sigma(\overline{T})$ of pairwise non-isomorphic indecomposables occurring as direct summands of \overline{T} . We start by estimating $\sigma(\overline{U})$ for an indecomposable representation U :

Lemma 3.3. *For an indecomposable representation $U \neq E$ of Q , we have*

$$\sigma(\overline{U}) \leq 1 + N(Q) \cdot \left(\left(\sum_{j=1}^s \dim U(y_j) \right) - \dim U(z) \right).$$

Proof. As U is indecomposable, we may assume Q to be connected. We use the following abbreviations:

$$\dim U(z) = u, \quad \dim U(y_j) = u_j, j = 1, \dots, s, \quad u' = \left(\sum_{j=1}^s u_j \right) - u.$$

Note that $u' \geq 0$ since U is indecomposable and $U \neq E$. If $u = 0$, $\bar{U} = U$ is indecomposable and $\sigma(\bar{U}) = 1$. In case $u' = 0$, the map

$$[U(\alpha_1), \dots, U(\alpha_s)] : \bigoplus_{j=1}^s U(y_j) \rightarrow U(z)$$

is an isomorphism, and again \bar{U} is indecomposable. Thus we may suppose $u > 0$ and $u' > 0$.

Recall that the value of the Tits form $q(\mathbf{dim} U)$ equals 0 or 1, as Q is tame. We compute:

$$\begin{aligned} q(\mathbf{dim} U - \mathbf{dim} E) &= q(\mathbf{dim} U) + q(\mathbf{dim} E) - \langle \mathbf{dim} U, \mathbf{dim} E \rangle - \langle \mathbf{dim} E, \mathbf{dim} U \rangle \\ &= q(\mathbf{dim} U) + q(\mathbf{dim} E) + u' - u \leq 2 + u' - u. \end{aligned}$$

As q is positive definite or positive semi-definite in case Q is a Dynkin quiver or an extended Dynkin quiver, respectively, we obtain:

$$u \leq \begin{cases} u' + 2 \leq 2u' + 1 & \text{if } Q \text{ is an extended Dynkin quiver,} \\ u' + 1 & \text{if } Q \text{ is a Dynkin quiver.} \end{cases}$$

Now clearly \bar{U} has at most $\sum_{j=1}^s u_j$ indecomposable direct summands, and thus

$$\sigma(\bar{U}) \leq \sum_{j=1}^s u_j = u + u' \leq \begin{cases} 1 + 3u' & \text{if } Q \text{ is an extended Dynkin quiver,} \\ 1 + 2u' & \text{if } Q \text{ is a Dynkin quiver,} \end{cases}$$

which proves the lemma except in case $|Q| = \mathbb{A}_n$ or $|Q| = \tilde{\mathbb{A}}_{n-1}$.

If $|Q| = \mathbb{A}_n$, we have $u \leq 1$ and hence $\sigma(\bar{U}) \leq 1 + u'$. In case $|Q| = \tilde{\mathbb{A}}_{n-1}$, the number of indecomposable (possible isomorphic) direct summands in a decomposition of \bar{U} is at most $1 + u'$. This can be seen by inspecting the list of indecomposable representations of Q . Such representations are string or band representations, and they are described by words (non-oriented paths) in Q (see [4] for details). \square

Proof of Corollary 3.2. We set

$$t'_i = \left(\sum_{j=1}^s \dim T_i(y_j) \right) - \dim T_i(z), \quad i = 1, \dots, r$$

and

$$t' = \sum_{i=1}^r t'_i.$$

Note that, by definition,

$$\sum_{i=1}^r \lambda_i t'_i = \left(\sum_{j=1}^s d_{y_j} \right) - d_z.$$

Our lemma implies:

$$\begin{aligned} \sigma(\bar{T}) &\leq \sum_{i=1}^r \sigma(\bar{T}_i) \leq r + N(Q) \cdot t' \leq r + \left(\sum_{i=1}^r \lambda_i t'_i \right) - (\lambda - N(Q)) \cdot t' \\ &= \sigma(T) + \left(\sum_{j=1}^s d_{y_j} \right) - d_z - (\lambda - N(Q)) \cdot t'. \end{aligned}$$

Combining this with Proposition 3.1 we find that

$$\begin{aligned} \text{codim } Z''_{\bar{Q}, \bar{\mathbf{d}}} - n + \sigma(T) &= \text{codim}_{\text{rep}(\bar{Q}, \bar{\mathbf{d}})} Z_{\bar{Q}, \bar{\mathbf{d}}} + \left(\sum_{j=1}^s d_{y_j} \right) - d_z + 1 - n + \sigma(T) \\ &\geq \text{codim}_{\text{rep}(\bar{Q}, \bar{\mathbf{d}})} Z_{\bar{Q}, \bar{\mathbf{d}}} - (n - 1) + \sigma(\bar{T}) + (\lambda - N(Q)) \cdot t'. \end{aligned}$$

As $t'_i \geq 0$ for all i , this yields part (i) of Corollary 3.2. Part (ii) follows from the fact that $\sum_{i=1}^r \lambda_i t'_i = \left(\sum_{j=1}^s d_{y_j} \right) - d_z > 0$ implies $t'_i > 0$ for some i and hence $t' > 0$. □

4. Reflection functors

We define two new quivers \tilde{Q} and Q' : \tilde{Q} is obtained from Q by adding a vertex z' and arrows $\beta_j : z' \rightarrow y_j, j = 1, \dots, s$. Deleting z and $\alpha_1, \dots, \alpha_s$ in \tilde{Q} yields Q' . Note that Q' is tame as well. We denote by E' the simple injective representation of Q' supported at z' .

We consider the reflection functor

$$\mathcal{F} : \text{rep}(Q) \rightarrow \text{rep}(Q')$$

associated with z . Recall that

$$(\mathcal{F}X)(i) = \begin{cases} X(i) & i \neq z' \\ \ker \left(\bigoplus X(y_j) \xrightarrow{[X(\alpha_1), \dots, X(\alpha_s)]} X(z) \right) & i = z', \end{cases}$$

and that

$$(\mathcal{F}X)(\beta_l) : (\mathcal{F}X)(z') \rightarrow (\mathcal{F}X)(y_l) = X(y_l)$$

is the inclusion of $(\mathcal{F}X)(z')$ into $\bigoplus_{j=1}^s X(y_j)$ followed by the projection to $X(y_l)$ (see [1], [6]). The functor \mathcal{F} restricts to an equivalence

$$\mathcal{F} : (\text{rep}(Q))' \rightarrow (\text{rep}(Q'))'$$

from the full subcategory $(\text{rep}(Q))'$ of $\text{rep}(Q)$ whose objects do not contain E as a direct summand, or equivalently have no non-trivial morphisms to E , to the full subcategory $(\text{rep}(Q'))'$ of $\text{rep}(Q')$ whose objects do not contain E' as a direct summand.

Suppose that E is neither a direct summand of T nor an element of T^\perp . This implies that $[T, E] = 0$ and ${}^1[T, E] > 0$ and thus the vector $\mathbf{d}' \in \mathbb{Z}^{Q'_0}$, where Q'_0 denotes the set of vertices of Q' , defined by

$$d'_x = \begin{cases} d_x, & x \neq z' \\ \left(\sum_{j=1}^s d_{y_j} \right) - d_z, & x = z' \end{cases}$$

has positive entries. Indeed, we have

$$d'_{z'} = \left(\sum_{j=1}^s d_{y_j} \right) - d_z = -\langle \mathbf{d}, \mathbf{dim} E \rangle = -[T, E] + {}^1[T, E] > 0. \tag{4.1}$$

Note that in fact we have $d'_{z'} \geq \lambda$ as ${}^1[T_i, E] > 0$ for some i implies ${}^1[T, E] \geq \lambda_i \geq \lambda$. We let $\tilde{\mathbf{d}}$ be the dimension vector for \tilde{Q} which coincides with \mathbf{d} on Q_0 and with \mathbf{d}' on Q'_0 .

As E is not a direct summand of T , the latter belongs to $(\text{rep } Q)'$. Therefore $\mathcal{F}T$ lies in $(\text{rep } Q')'$, and we have $\mathbf{dim} \mathcal{F}T = \mathbf{d}'$, ${}^1[\mathcal{F}T, \mathcal{F}T] = {}^1[T, T] = 0$, and thus \mathbf{d}' is prehomogeneous. Choose T' in the open orbit of $\text{rep}(Q', \mathbf{d}')$. As T' is isomorphic to $\mathcal{F}T$, we have $T' = \bigoplus_{i=1}^r (T'_i)^{\lambda_i}$ with T'_i indecomposable, pairwise non-isomorphic and ${}^1[T'_i, T'_j] = 0$ for all i, j . Moreover, we know $T^\perp \subseteq (\text{rep } Q)'$, as E does not belong to T^\perp , and $(T')^\perp \subseteq (\text{rep } Q')'$, as $d'_{z'} = [T', E] > 0$. We conclude that $(T')^\perp$ is equivalent to $\mathcal{F}(T^\perp)$, the category of representations of a quiver with $n - r$ vertices. Hence $\mathcal{Z}_{Q', \mathbf{d}'}$ is given by $n - r$ equations as well. We decompose $\mathcal{Z}_{Q', \mathbf{d}'}$ as a disjoint union $\mathcal{Z}_{Q', \mathbf{d}'} = \mathcal{W}'_{Q', \mathbf{d}'} \cup \mathcal{W}''_{Q', \mathbf{d}'}$, where

$$\mathcal{W}'_{Q', \mathbf{d}'} = \{X' \in \mathcal{Z}_{Q', \mathbf{d}'}; [E', X'] = 0\}$$

and

$$\mathcal{W}''_{Q', \mathbf{d}'} = \{X' \in \mathcal{Z}_{Q', \mathbf{d}'}; [E', X'] \neq 0\}.$$

Proposition 4.1. *Suppose E is neither a direct summand of T nor an element of T^\perp . Then we have*

(i) $\text{codim } \mathcal{Z}'_{Q, \mathbf{d}} = \text{codim}_{\text{rep}(Q', \mathbf{d}')} \mathcal{W}'_{Q', \mathbf{d}'}$

and

(ii) $\mathcal{Z}'_{Q, \mathbf{d}}$ is irreducible if $\mathcal{W}'_{Q', \mathbf{d}'}$ has this property.

Proof. By construction, X belongs to $\mathcal{Z}'_{Q, \mathbf{d}}$ if and only if $\mathcal{F}X$ is isomorphic to some $X' \in \mathcal{W}'_{Q', \mathbf{d}'}$, but unfortunately the functor \mathcal{F} cannot be made into a regular map from

$$\text{rep}(Q, \mathbf{d})' = \{X \in \text{rep}(Q, \mathbf{d}); [X, E] = 0\}$$

to

$$\text{rep}(Q', \mathbf{d}') = \{X' \in \text{rep}(Q', \mathbf{d}'); [E', X'] = 0\}.$$

We use the following détour (compare [7] and Section 4.2 in [3]): The set

$$\left\{ X \in \text{rep}(\tilde{Q}, \tilde{\mathbf{d}}); \sum_{j=1}^s X(\alpha_j)X(\beta_j) = 0, [X(\beta_1), \dots, X(\beta_s)]^t \text{ injective}, \right. \\ \left. [X(\alpha_1), \dots, X(\alpha_s)] \text{ surjective} \right\}$$

is a principal $\text{Gl}(d'_z)$ -bundle over $\text{rep}(Q, \mathbf{d})'$ and a principal $\text{Gl}(d_z)$ -bundle over $\text{rep}(Q', \mathbf{d})'$ via the projections π and π' deleting z' and z , respectively. Hence the claim follows from $\pi^{-1}(\mathcal{Z}'_{Q, \mathbf{d}}) = (\pi')^{-1}(\mathcal{W}'_{Q', \mathbf{d}'})$. \square

5. Proof of Theorem 1.1

We proceed by induction on the number n of vertices of Q . We may assume the theorem to be true for $\mathcal{Z}_{Q, \mathbf{d}}$. First we treat the cases that

- (i) E is a direct summand of T

and

- (ii) E belongs to T^\perp .

In both cases, we have that E is a direct summand of X for all $X \in \mathcal{Z}_{Q, \mathbf{d}}$; i.e., $\mathcal{Z}''_{Q, \mathbf{d}} = \mathcal{Z}_{Q, \mathbf{d}}$. Indeed, in case (i) this follows from the fact that $\text{Hom}_Q(E, T) \neq 0$, which is a closed condition. In case (ii), E is a simple object in T^\perp .

As any direct summand $T_i \not\cong E$ of T belongs to ${}^\perp E$, we have

$$\dim T_i(z) - \sum_{j=1}^s \dim T_i(y_j) = \langle \mathbf{dim} T_i, \mathbf{dim} E \rangle = [T_i, E] - {}^1[T_i, E] = 0.$$

By Lemma 3.3, \overline{T}_i is indecomposable, and therefore

$$\sigma(\overline{T}) = \begin{cases} r - 1 & \text{in case (i),} \\ r & \text{in case (ii).} \end{cases}$$

The induction hypothesis together with Corollary 3.2 implies the first part of our theorem. We conclude from Proposition 3.1 that $\mathcal{Z}_{Q, \mathbf{d}} \simeq \mathcal{Z}_{\overline{Q}, \overline{\mathbf{d}}} \times \mathcal{N}_{\mathbf{d}}$, where

$$\mathcal{N}_{\mathbf{d}} = \{A \in \text{Mat} \left(d_z \times \sum_{j=1}^s d_{y_j} \right); \text{rank } A < d_z\}.$$

The second part follows from the fact that the set $\mathcal{N}_{\mathbf{d}}$ is irreducible in case $d_z \geq \sum_{j=1}^s d_{y_j}$.

- (iii) Finally, suppose that E is neither a direct summand of T nor does it belong to T^\perp , or equivalently that $d_z < \sum_{j=1}^s d_{y_j}$. Using Corollary 3.2 and its dual, Proposition 4.1 and remembering that the codimension of any irreducible

component of $\mathcal{Z}_{Q,\mathbf{d}}$ is at most $n - r$, we see that the theorem is true for $\mathcal{Z}_{Q,\mathbf{d}}$ if and only if it holds for $\mathcal{Z}_{Q',\mathbf{d}'}$.

In case either T contains a preprojective direct summand or T^\perp a preprojective representation, we may apply a series of reflection functors until we reach the situation that a simple projective either is a direct summand of T or else belongs to T^\perp , and we can reduce by (i) or (ii). This finishes the proof in case Q is of finite representation type as any indecomposable representation is preprojective.

If Q is not representation finite, we are left with the situation that no preprojective representation is a direct summand of T nor an element of T^\perp . Dually, we may assume T does not contain a preinjective direct summand either. Indeed, suppose a simple injective representation E' is a direct summand of T or belongs to ${}^\perp T$, a situation we will reach after a series of (inverse) reflection functors. Then apply the dual of the first or the second reduction step above; recall that $\mathcal{Z}_{Q,\mathbf{d}}$ has a dual description as

$$\mathcal{Z}_{Q,\mathbf{d}} = \{X \in \text{rep}(Q, \mathbf{d}); [S', X] \neq 0 \text{ for all simple objects } S' \in {}^\perp T\}.$$

The following lemma finishes the proof of Theorem 1.1.

Lemma 5.1. *Let Q be an extended Dynkin quiver. Suppose T is a regular representation with an open orbit. Then T^\perp contains a non-zero preprojective representation.*

Proof. Consider a Bongartz completion \tilde{T} for T [2]; i.e., an exact sequence

$$0 \rightarrow kQ \rightarrow \tilde{T} \rightarrow \bigoplus_{i=1}^r T_i^{\nu_i} \rightarrow 0$$

for which the induced map

$$\text{Hom}_Q \left(T_l, \bigoplus_{i=1}^r T_i^{\nu_i} \right) \rightarrow \text{Ext}_Q^1(T_l, kQ)$$

is surjective for $l = 1, \dots, r$. There is a \mathbb{Z} -linear map $\partial : \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$, called defect, such that any indecomposable representation Y of Q is preprojective, regular and preinjective if and only if $\partial(\mathbf{dim} Y)$ is negative, zero and positive, respectively (see for instance [10]). As T is regular, $\partial \tilde{T} = \partial kQ < 0$ and therefore \tilde{T} contains an indecomposable preprojective direct summand Y , and $Y \in T^\perp$. Indeed, we have ${}^1[T, Y] = 0$ for all direct summands of \tilde{T} and $[T, Y] = 0$ since Y is preprojective and T is regular [10, Theorem 3.6.5]. □

Example. Working out the following example, one can see that if Q is not tame, it may happen that both T and T^\perp belong to the set of regular representations:



