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On Serre’s injectivity question and norm principle

Nivedita Bhaskhar*

Abstract. Let k be a field of characteristic not 2. We give a positive answer to Serre’s injectivity question for any smooth connected reductive k -group whose Dynkin diagram contains connected components only of type A_n , B_n or C_n . We do this by relating Serre’s question to the norm principles proved by Barquero and Merkurjev. We give a scalar obstruction defined up to spinor norms whose vanishing will imply the norm principle for the non-trialitarian D_n case and yield a positive answer to Serre’s question for connected reductive k -groups whose Dynkin diagrams contain components of (non-trialitarian) type D_n too. We also investigate Serre’s question for quasi-split reductive k -groups.

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Keywords. Principal homogeneous spaces, Serre’s question, zero cycles, norm principles, spinor norm, Galois cohomology.

1. Introduction

Let k be a field. Then the following question of Serre, which is open in general, asks

Question 1.1 (Serre, [13, p. 233]). *Let G be any connected linear algebraic group over a field k . Let L_1, L_2, \dots, L_r be finite field extensions of k of degrees d_1, d_2, \dots, d_r respectively such that $\gcd_i(d_i) = 1$. Then is the following sequence exact ?*

$$1 \rightarrow H^1(k, G) \rightarrow \prod_{i=1}^r H^1(L_i, G).$$

The classical result that the index of a central simple algebra divides the degrees of its splitting fields answers Serre’s question affirmatively for the group PGL_n . Springer’s theorem for quadratic forms answers it affirmatively for the (albeit sometimes disconnected) group $\mathrm{O}(q)$ and Bayer–Lenstra’s theorem [2], for the groups of isometries of algebras with involutions. Jodi Black [3] answers Serre’s question positively for absolutely simple simply connected and adjoint k -groups of classical type. In this paper, we use and extend Jodi’s result to connected reductive k -groups whose Dynkin diagram contains connected components only of type A_n , B_n or C_n .

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Theorem 1.2. *Let k be a field of characteristic not 2. Let G be a connected reductive k -group whose Dynkin diagram contains connected components only of type A_n , B_n or C_n . Then Serre's question has a positive answer for G .*

We also investigate Serre's question for reductive k -groups whose derived subgroups admit quasi-split simply connected covers. More precisely, we give a uniform proof for the following :

Theorem 1.3. *Let k be a field of characteristic not 2. Let G be a connected quasi-split reductive k -group whose Dynkin diagram does not contain connected components of type E_8 . Then Serre's question has a positive answer for G .*

We relate Serre's question for G with the norm principles of other closely related groups following a series of reductions previously used by Barquero and Merkurjev to prove the norm principles for reductive groups whose Dynkin diagrams do not contain connected components of type D_n , E_6 or E_7 [1]. We also give a scalar obstruction defined up to spinor norms whose vanishing will imply the norm principle for the (non-trialitarian) D_n case and yield a positive answer to Serre's question for connected reductive k -groups whose Dynkin diagrams contain components of this type also.

In the next section, we begin with some lemmata and preliminary reductions. In Section 3, we introduce intermediate groups \hat{G} and \tilde{G} and relate Serre's question for G to Serre's question for \hat{G} and \tilde{G} via the norm principle. In Section 4, we investigate the norm principle for (non-trialitarian) type D_n groups and find the scalar obstruction whose vanishing will imply the norm principle for the (non-trialitarian) D_n case. In the final section, we use the reduction techniques used in Sections 2 and 3 to discuss Serre's question for connected reductive k -groups whose derived subgroups admit quasi-split simply connected covers.

2. Preliminaries

We work over the base field k of characteristic not 2. By a k -group, we mean a smooth connected linear algebraic group defined over k . And mostly, we will restrict ourselves to reductive groups. We say that a k -group G satisfies SQ if Serre's question has a positive answer for G .

2.1. Reduction to characteristic 0. Let G be a connected reductive k -group whose Dynkin diagram contains connected components only of type A_n , B_n , C_n or (non-trialitarian) D_n . Without loss of generality we may assume that k is of characteristic 0 [7, p. 47]. We give a sketch of the reduction argument for the sake of completeness.

Suppose that the characteristic of k is $p > 0$. Let L_1, L_2, \dots, L_r be finite field extensions of k of degrees d_1, d_2, \dots, d_r respectively such that $\gcd_i(d_i) = 1$ and

let ξ be an element in the kernel of

$$H^1(k, G) \rightarrow \prod_{i=1}^r H^1(L_i, G).$$

By a theorem of Gabber, Liu and Lorenzini [5, Thm. 9.2] which was pointed out to us by O. Wittenberg, we note that any torsor under a smooth group scheme G/k which admits a zero-cycle of degree 1 also admits a zero-cycle of degree 1 whose support is étale over k . Thus without loss of generality we can assume that the given coprime extensions L_i/k are in fact separable.

By [10, Thms. 1 & 2], there exists a complete discrete valuation ring R with residue field k and fraction field K of characteristic zero. Let S_i denote corresponding étale extensions of R with residue fields L_i and fraction fields K_i .

There exists a smooth R -group scheme \tilde{G} with special fiber G and connected reductive generic fiber \tilde{G}_K . Now given any torsor $t \in H^1(k, G)$, there exists a torsor $\tilde{t} \in H^1_{\text{ét}}(R, \tilde{G})$ specializing to t which is unique upto isomorphism. This in turn gives a torsor \tilde{t}_K in $H^1(K, \tilde{G}_K)$ by base change, thus defining a map $i_k : H^1(k, G) \rightarrow H^1(K, \tilde{G}_K)$ [6, p. 29]. It clearly sends the trivial element to the trivial element. The map i also behaves well with the natural restriction maps, i.e., it fits into the following commutative diagram :

$$\begin{CD} H^1(k, G) @>i_k>> H^1(K, \tilde{G}_K) \\ @VVV @VVV \\ \prod H^1(L_i, G) @>\prod i_{L_i}>> \prod H^1(K_i, \tilde{G}_K). \end{CD}$$

Let $\tilde{\xi}$ denote the torsor in $H^1_{\text{ét}}(R, \tilde{G})$ corresponding to ξ as above. Therefore $\tilde{\xi}_K := i_k(\xi)$ is in the kernel of

$$H^1(K, \tilde{G}_K) \rightarrow \prod_{i=1}^r H^1(K_i, \tilde{G}_K).$$

Suppose that \tilde{G}_K satisfies SQ . Then $\tilde{\xi}_K$ is trivial. However by [12], the natural map $H^1_{\text{ét}}(R, \tilde{G}) \rightarrow H^1(K, \tilde{G}_K)$ is injective and hence $\tilde{\xi}$ is trivial in $H^1_{\text{ét}}(R, \tilde{G})$. This implies that its specialization, ξ , is trivial in $H^1(k, G)$.

Thus from here on, we assume that the base field k has characteristic 0.

2.2. Lemmata.

Lemma 2.1. *Let k -groups G and H satisfy SQ . Then $G \times_k H$ also satisfies SQ .*

Proof. Let L/k be a field extension. Then the map

$$H^1(k, G \times_k H) \rightarrow H^1(L, G \times_k H)$$

is precisely the product of the maps

$$H^1(k, G) \rightarrow H^1(L, G) \text{ and } H^1(k, H) \rightarrow H^1(L, H).$$

This immediately shows that if G and H satisfy SQ , so does $G \times_k H$. □

Lemma 2.2. *Let $1 \rightarrow Q \rightarrow H \rightarrow G \rightarrow 1$ be a central extension of a k -group G by a quasi-trivial torus Q . Then H satisfies SQ if and only if G satisfies SQ .*

Proof. Let L_i be field extensions of k such that $\gcd[L_i : k] = 1$. Since Q is quasi-trivial, $H^1(L, Q) = \{1\} \forall L/k$. From the long exact sequence in cohomology, we have the following commutative diagram.

$$\begin{array}{ccccccc} 1 & \longrightarrow & H^1(k, H) & \longrightarrow & H^1(k, G) & \xrightarrow{\delta_k} & H^2(k, Q) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \prod H^1(L_i, H) & \longrightarrow & \prod H^1(L_i, G) & \xrightarrow{\prod \delta_{L_i}} & \prod H^2(L_i, Q) \end{array}$$

From the above diagram, it is clear that if G satisfies SQ , so does H .

Conversely, assume that H satisfies SQ . Let $a \in H^1(k, G)$ become trivial in $\prod H^1(L_i, G)$. Then $\delta_k(a)$ becomes trivial in each $H^2(L_i, Q)$. Hence the corestriction $\text{Cor}_{L_i/k}(\delta_k(a)) = \delta_k(a)^{d_i}$ becomes trivial in $H^2(k, Q)$ where $d_i = [L_i : k]$. Since $\gcd_i(d_i) = 1$, this implies that $\delta_k(a)$ is itself trivial in $H^2(k, Q)$. Therefore a comes from an element $b \in H^1(k, H)$ which is trivial in $\prod H^1(L_i, H)$. (The fact that $H^1(L_i, Q) = \{1\}$ guarantees that b is trivial in $H^1(L_i, H)$.) Since H satisfies SQ by assumption, b is trivial in $H^1(k, H)$ which implies the triviality of a in $H^1(k, G)$. □

Lemma 2.3. *Let E be a finite separable field extension of k and let H be an E -group satisfying SQ . Then the k -group $R_{E/k}(H)$ also satisfies SQ .*

Proof. Set $G = R_{E/k}(H)$ and let ξ be an element in the kernel of $H^1(k, G) \rightarrow \prod_{i=1}^r H^1(L_i, G)$ where $\gcd_i [L_i : k] = 1$.

Since $\text{char}(k) = 0$, $L_i \otimes_k E$ is an étale E -algebra and hence isomorphic to $E_{1,i} \times E_{2,i} \times \dots \times E_{n_i,i}$ where each $E_{j,i}$ is a separable field extension of E . Thus $\sum_{j=1}^{n_i} [E_{j,i} : E] = [L_i : k]$ and therefore $\gcd [E_{j,i} : E] = 1$ where $1 \leq i \leq r$ and $1 \leq j \leq n_i$.

By Eckmann–Faddeev–Shapiro, we have a natural bijection of pointed sets

$$\begin{aligned} H^1(k, G) &\simeq H^1(E, H), \\ H^1(L_i, G) &\simeq \prod_{j=1}^{n_i} H^1(E_{j,i}, H). \end{aligned}$$

Thus we have that ξ is in the kernel of $H^1(E, H) \rightarrow \prod_{i \leq r, j \leq n_i} H^1(E_{j,i}, H)$. Since H satisfies SQ , we see that ξ is trivial. □

3. Serre’s question and norm principles

3.1. Intermediate groups \hat{G} and \tilde{G} . *Notations are as in Section 5 of [1].*

Let G be our given connected reductive k -group whose Dynkin diagram contains connected components only of type A_n, B_n, C_n or (non-trivialitarian) D_n and let G' denote its derived subgroup. Let $Z(G) = T$ and $Z(G') = \mu$.

Let $\rho : \mu \hookrightarrow S$ be an embedding of μ into a quasi-trivial torus S . We denote the cofibre product $e(G', \rho) = \frac{G' \times S}{\mu}$ by \hat{G} . This k -group is called an *envelope* of G' .

$$\begin{array}{ccc} \mu & \xrightarrow{\delta} & G' \\ \downarrow \rho & & \downarrow \\ S & \xrightarrow{\gamma} & \hat{G} \end{array}$$

Now the quasi-trivial torus $S = Z(\hat{G})$ and \hat{G} fit into an exact sequence as follows:

$$1 \rightarrow S \rightarrow \hat{G} \rightarrow G'^{ad} \rightarrow 1 \tag{*}$$

where G'^{ad} corresponds to the adjoint group of G' . We now recall the following result of Jodi Black which addresses Serre’s question for adjoint groups of classical type.

Theorem 3.1 (Jodi Black, [3, Thm. 0.2]). *Let k be a field of characteristic different from 2 and let J be an absolutely simple algebraic k -group which is not of type E_8 and which is either a simply connected or adjoint classical group or a quasi-split exceptional group. Then Serre’s question has a positive answer for J .*

Since every adjoint group of classical type is a product of Weil restrictions of absolutely simple adjoint groups, the above theorem, along with Lemmata 2.1 and 2.3, implies that G'^{ad} satisfies SQ . Applying Lemma 2.2 to the exact sequence (*) above, we see that \hat{G} satisfies SQ . *Let us chose such an envelope \hat{G} of G' which satisfies SQ .*

Define an intermediate abelian group \tilde{T} to be the cofibre product $\frac{T \times S}{\mu}$.

$$\begin{array}{ccc} \mu & \longrightarrow & T \\ \downarrow \rho & & \downarrow \alpha \\ S & \xrightarrow{\nu} & \tilde{T} \end{array}$$

Let the algebraic group \tilde{G} be the cofibre product defined by the following diagram:

$$\begin{array}{ccc} G' \times T & \xrightarrow{m} & G \\ \downarrow id \times \alpha & & \downarrow \beta \\ G' \times \tilde{T} & \xrightarrow{\epsilon} & \tilde{G}. \end{array}$$

Then we have the following commutative diagram with exact rows [1, Prop. 5.1]. Note that each row is a central extension of \tilde{G} .

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \mu & \xrightarrow{\delta, \nu\rho} & G' \times \tilde{T} & \xrightarrow{\epsilon} & \tilde{G} \longrightarrow 1 \\
 & & \downarrow \rho & & \downarrow & & \downarrow id \\
 1 & \longrightarrow & S & \xrightarrow{\gamma, \nu} & \hat{G} \times \tilde{T} & \longrightarrow & \tilde{G} \longrightarrow 1
 \end{array}
 \quad \begin{array}{l} (**) \\ (***) \end{array}$$

Since \tilde{T} is abelian, the existence of the co-restriction map shows that \tilde{T} satisfies SQ . Since \hat{G} satisfies SQ , we can apply Lemmata 2.1 and 2.2 to (***) to see that \tilde{G} satisfies SQ .

3.2. Norm principle and weak norm principle. Let $f : G \rightarrow T$ be a map of k -groups where T is an abelian k -group. Then we have norm maps $N_{L/k} : T(L) \rightarrow T(k)$ for any separable field extension L/k .

$$\begin{array}{ccc}
 G(L) & \xrightarrow{f(L)} & T(L) \\
 & & \downarrow N_{L/k} \\
 G(k) & \xrightarrow{f(k)} & T(k)
 \end{array}$$

We say that the *norm principle* holds for $f : G \rightarrow T$ if for all separable field extensions L/k ,

$$N_{L/k}(\text{Image } f(L)) \subseteq \text{Image } f(k).$$

That is, we say that the *norm principle* holds for $f : G \rightarrow T$ if given any separable field extension L/k and any $t \in T(L)$ such that

$$t \in (\text{Image } f(L) : G(L) \rightarrow T(L)),$$

then

$$N_{L/k}(t) \in (\text{Image } f(k) : G(k) \rightarrow T(k)).$$

Note that the norm principle holds for any algebraic group homomorphism between abelian groups.

We say that the *weak norm principle* holds for $f : G \rightarrow T$ if given any $t \in T(k)$ such that

$$t \in (\text{Image } f(L) : G(L) \rightarrow T(L)),$$

then

$$t^{[L:k]} = N_{L/k}(t) \in (\text{Image } f(k) : G(k) \rightarrow T(k)).$$

It is clear that if the norm principle holds for f , then so does the weak norm principle.

3.3. Relating Serre’s question and norm principle. The deduction of SQ for G from \hat{G} and \tilde{G} follows via the (weak) norm principles.

Let $\beta : G \rightarrow \tilde{G}$ be the embedding of k -groups with the cokernel P isomorphic to the torus $\frac{S}{\mu}$ where \tilde{G} and G are as in Section 3.1. Thus we have the following exact sequence:

$$1 \rightarrow G \xrightarrow{\beta} \tilde{G} \xrightarrow{\pi} P \rightarrow 1.$$

Lemma 3.2. *If the weak norm principle holds for $\pi : \tilde{G} \rightarrow P$, then G satisfies SQ.*

Proof. From the long exact sequence of cohomology, we have the following commutative diagram:

$$\begin{array}{ccccccccccc} 1 & \rightarrow & G(k) & \rightarrow & \tilde{G}(k) & \xrightarrow{\pi_k} & P(k) & \xrightarrow{\delta_k} & H^1(k, G) & \xrightarrow{\beta_k} & H^1(k, \tilde{G}) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \rightarrow & \prod G(L_i) & \rightarrow & \prod \tilde{G}(L_i) & \xrightarrow{\prod \pi_{L_i}} & \prod P(L_i) & \xrightarrow{\prod \delta_{L_i}} & \prod H^1(L_i, G) & \rightarrow & \prod H^1(L_i, \tilde{G}). \end{array}$$

Let $a \in H^1(k, G)$ become trivial in $\prod H^1(L_i, G)$. As \tilde{G} satisfies SQ, $\beta_k(a)$ becomes trivial in $H^1(k, \tilde{G})$. Hence $a = \delta_k(b)$ for some $b \in P(k)$ and $\delta_{L_i}(b)$ is trivial in $H^1(L_i, G)$. Therefore, there exist $c_i \in \tilde{G}(L_i)$ such that $\pi_{L_i}(c_i) = b$.

Showing that G satisfies SQ, i.e. that a is trivial, is equivalent to showing

$$b \in (\text{Image } \pi_k : \tilde{G}(k) \rightarrow P(k)).$$

However $b \in (\text{Image } \pi_{L_i} : \tilde{G}(L_i) \rightarrow P(L_i))$. Since the weak norm principle holds for $\pi : \tilde{G} \rightarrow P$, $b^{d_i} \in \text{Image } (\pi_k : \tilde{G}(k) \rightarrow P(k))$ where $[L_i : k] = d_i$ for each i . As $\text{gcd}_i(d_i) = 1$, this means $b \in \text{Image } (\pi_k : \tilde{G}(k) \rightarrow P(k))$. □

We recall now the norm principle of Merkurjev and Barquero for reductive groups of classical type.

Theorem 3.3 (Barquero–Merkurjev, [1]). *Let G be a reductive group over a field k . Assume that the Dynkin diagram of G does not contain connected components D_n , $n \geq 4$, E_6 or E_7 . Let T be any commutative k -group. Then the norm principle holds for any group homomorphism $G \rightarrow T$.*

This shows that the norm principle and hence the weak norm principle holds for the map $\pi : \tilde{G} \rightarrow P$ for reductive k -groups G as in the main theorem (Theorem 1.2). Thus we have concluded the proof for the following:

Theorem 1.2. *Let k be a field of characteristic not 2. Let G be a connected reductive k -group whose Dynkin diagram contains connected components only of type A_n , B_n or C_n . Then Serre’s question has a positive answer for G .*

4. Obstruction to norm principle for (non-trialitarian) D_n

4.1. Preliminaries. Let (A, σ) be a central simple algebra of degree $2n$ over k and let σ be an orthogonal involution. Let $C(A, \sigma)$ denote its Clifford algebra which is a central simple algebra over its center, Z/k , the discriminant extension. Let i denote the non-trivial automorphism of Z/k and let $\underline{\sigma}$ denote the canonical involution of $C(A, \sigma)$.

Recall that, depending on the parity of n , $\underline{\sigma}$ is either an involution of the second kind (when n is odd) or of the first kind (when n is even). Let $\underline{\mu} : \text{Sim}(C(A, \sigma), \underline{\sigma}) \rightarrow R_{Z/k} \mathbb{G}_m$ denote the multiplier map sending similitude c to $\underline{\sigma}(c)c$.

Let $\Omega(A, \sigma)$ be the *extended Clifford group*. Note that this has center $R_{Z/k} \mathbb{G}_m$ and is an *envelope* of $\text{Spin}(A, \sigma)$ [1, Ex. 4.4]. We recall below the map $\varkappa : \Omega(A, \sigma)(k) \rightarrow Z^*/k^*$ as defined in [9, p. 182].

Given $\omega \in \Omega(A, \sigma)(k)$, let $g \in \text{GO}^+(A, \sigma)(k)$ be some similitude such that $\omega \rightsquigarrow gk^*$ under the natural surjection $\Omega(A, \sigma)(k) \rightarrow \text{PGO}^+(A, \sigma)(k)$.

Let $h = \mu(g)^{-1}g^2 \in \text{O}^+(A, \sigma)(k)$ and let $\gamma \in \Gamma(A, \sigma)(k)$ be some element in the *special Clifford group* which maps to h under the vector representation $\chi' : \Gamma(A, \sigma)(k) \rightarrow \text{O}^+(A, \sigma)(k)$. Then $\omega^2 = \gamma z$ for some $z \in Z^*$ and $\varkappa(\omega) = zk^*$.

Note that the map \varkappa has $\Gamma(A, \sigma)(k)$ as kernel. Also if $z \in Z^*$, then $\varkappa(z) = z^2k^*$.

By following the reductions in [1], it is easy to see that one needs to investigate whether the norm principle holds for the canonical map

$$\Omega(A, \sigma) \rightarrow \frac{\Omega(A, \sigma)}{[\Omega(A, \sigma), \Omega(A, \sigma)]}.$$

We will need to investigate the norm principle for two different maps depending on the parity of n .

The map μ_* for n odd. Let $U \subset \mathbb{G}_m \times R_{Z/k} \mathbb{G}_m$ be the algebraic subgroup defined by

$$U(k) = \{(f, z) \in k^* \times Z^* \mid f^4 = N_{Z/k}(z)\}.$$

Recall the map $\mu_* : \Omega(A, \sigma) \rightarrow U$ defined in [9, p. 188] which sends

$$\omega \rightsquigarrow \left(\underline{\mu}(\omega), ai(a)^{-1} \underline{\mu}(\omega)^2 \right),$$

where $\omega \in \Omega(A, \sigma)(k)$ and $\varkappa(\omega) = a k^*$. This induces the following exact sequence [9, p. 190]

$$1 \rightarrow \text{Spin}(A, \sigma) \rightarrow \Omega(A, \sigma) \xrightarrow{\mu_*} U \rightarrow 1.$$

Since the semisimple part of $\Omega(A, \sigma)$ is $\text{Spin}(A, \sigma)$, the above exact sequence shows that it suffices to check the norm principle for the map μ_* .

The map $\underline{\mu}$ for n even. Recall the following exact sequence induced by restricting $\underline{\mu}$ to $\Omega(A, \sigma)$ [9, p. 187]

$$1 \rightarrow \text{Spin}(A, \sigma) \rightarrow \Omega(A, \sigma) \xrightarrow{\underline{\mu}} R_{Z/k} \mathbb{G}_m \rightarrow 1.$$

Since the semisimple part of $\Omega(A, \sigma)$ is $\text{Spin}(A, \sigma)$, the above exact sequence shows that it suffices to check the norm principle for the map $\underline{\mu}$.

4.2. An obstruction to being in the image of μ_* for n odd. Given $(f, z) \in U(k)$, we would like to formulate an obstruction which prevents (f, z) from being in the image $\mu_*(\Omega(A, \sigma)(k))$. Note that for $z \in Z^*$, $\mu_*(z) = (N_{Z/k}(z), z^4)$ and hence the algebraic subgroup $U_0 \subseteq U$ defined by

$$U_0(k) = \{(N_{Z/k}(z), z^4) | z \in Z^*\}$$

has its k -points in the image $\mu_*(\Omega(A, \sigma)(k))$.

Let $\mu_{n[Z]}$ denote the kernel of the norm map $R_{K/k} \mu_n \xrightarrow{N} \mu_n$ where K/k is a quadratic extension. Note that $\mu_{4[Z]}$ is the center of $\text{Spin}(A, \sigma)$ as n is odd. Also recall that [9, Prop. 30.13, p. 418]

$$H^1(k, \mu_{4[Z]}) \cong \frac{U(k)}{U_0(k)}.$$

Thus, we can construct the map $S : \text{PGO}^+(A, \sigma)(k) \rightarrow H^1(k, \mu_{4[Z]})$ induced by the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 1 & \longrightarrow & Z^* & \longrightarrow & \Omega(A, \sigma)(k) & \xrightarrow{\chi'} & \text{PGO}^+(A, \sigma)(k) & \longrightarrow & 1 \\ & & \downarrow \mu_* & & \downarrow \mu_* & & \downarrow S & & \\ 1 & \longrightarrow & U_0(k) & \longrightarrow & U(k) & \longrightarrow & H^1(k, \mu_{4[Z]}) & \longrightarrow & 1 \end{array}$$

The map S also turns out to be the connecting map from $\text{PGO}^+(A, \sigma)(k) \rightarrow H^1(k, \mu_{4[Z]})$ [9, Prop. 13.37, p. 190] in the long exact sequence of cohomology corresponding to the exact sequence

$$1 \rightarrow \mu_{4[Z]} \rightarrow \text{Spin}(A, \sigma) \rightarrow \text{PGO}^+(A, \sigma) \rightarrow 1.$$

Since the maps $\mu_* : Z^* \rightarrow U_0(k)$ and $\chi' : \Omega(A, \sigma)(k) \rightarrow \text{PGO}^+(A, \sigma)(k)$ are surjective, an element $(f, z) \in U(k)$ is in the image $\mu_*(\Omega(A, \sigma)(k))$ if and only if its image $[f, z] \in H^1(k, \mu_{4[Z]})$ is in the image $S(\text{PGO}^+(A, \sigma)(k))$.

Therefore we look for an obstruction preventing $[f, z]$ from being in the image $S(\text{PGO}^+(A, \sigma)(k))$. Recall the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & & & & & 1 \\
 & & & & & & \downarrow \\
 & & & & & & \mu_2 \\
 & & & & & & \downarrow \\
 1 & \longrightarrow & \mu_2 & \longrightarrow & \text{Spin}(A, \sigma) & \xrightarrow{\chi} & \text{O}^+(A, \sigma) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow id & & \downarrow \pi & & \\
 1 & \longrightarrow & \mu_{4[Z]} & \longrightarrow & \text{Spin}(A, \sigma) & \xrightarrow{\chi'} & \text{PGO}^+(A, \sigma) & \longrightarrow & 1 \\
 & & & & & & \downarrow & & \\
 & & & & & & 1 & &
 \end{array}$$

The long exact sequence of cohomology induces the following commutative diagram (Figure 1) with exact columns [9, Prop. 13.36, p. 189], where

$$\begin{array}{ccc}
 \text{O}^+(A, \sigma)(k) & \xrightarrow{S_n} & \frac{k^*}{k^{*2}} \\
 \downarrow \pi & & \downarrow i \\
 \text{PGO}^+(A, \sigma)(k) & \xrightarrow{S} & \text{H}^1(k, \mu_{4[Z]}) \\
 \downarrow \mu & & \downarrow j \\
 \frac{k^*}{k^{*2}} & = & \frac{k^*}{k^{*2}}
 \end{array}$$

Figure 1. Spinor norms and S for n odd

$\mu : \text{PGO}^+(A, \sigma)(k) \rightarrow \frac{k^*}{k^{*2}}$ is induced by the multiplier map $\mu : \text{GO}^+(A, \sigma) \rightarrow \mathbb{G}_m$

$i : \frac{k^*}{k^{*2}} \rightarrow \text{H}^1(k, \mu_{4[Z]}) = \frac{U(k)}{U_0(k)}$ is the map sending $f k^{*2} \rightsquigarrow [f, f^2]$

$j : \frac{U(k)}{U_0(k)} = \text{H}^1(k, \mu_{4[Z]}) \rightarrow \frac{k^*}{k^{*2}}$ is the map sending $[f, z] \rightsquigarrow \text{N}(z_0)k^{*2}$,

where $z_0 \in Z^*$ is such that $z_0 i(z_0)^{-1} = f^{-2}z$.

Definition 4.1. We call an element $(f, z) \in U(k)$ to be *special* if there exists a $[g] \in \text{PGO}^+(A, \sigma)(k)$ such that $j([f, z]) = \mu([g])$.

Let $(f, z) \in U(k)$ be a special element and let $[g] \in \text{PGO}^+(A, \sigma)(k)$ be such that $j([f, z]) = \mu([g])$. From the discussion above, it is clear that (f, z) is in the image $\mu_*(\Omega(A, \sigma)(k))$ if and only if $[f, z]$ is in the image $S(\text{PGO}^+(A, \sigma)(k))$.

Thus $S([g])[f, z]^{-1}$ is in kernel $j = \text{Image } i$ and hence there exists some $\alpha \in k^*$ such that

$$[f, z] = S([g])[\alpha, \alpha^2] \in \frac{U(k)}{U_0(k)}.$$

Note that if g is changed by an element in $O^+(A, \sigma)(k)$, then α changes by a spinor norm by Figure 1 above. Thus given a special element, we have produced a scalar $\alpha \in k^*$ which is well defined upto spinor norms.

$$\begin{aligned} [f, z] \in S(\text{PGO}^+(A, \sigma)(k)) &\iff [\alpha, \alpha^2] \in S(\text{PGO}^+(A, \sigma)(k)) \\ &\iff (\alpha, \alpha^2) \in \mu_*(\Omega(A, \sigma)(k)). \end{aligned}$$

This happens if and only if there exists $w \in \Omega(A, \sigma)(k)$ such that

$$\begin{aligned} \alpha &= \underline{\mu}(w) \\ \alpha^2 &= \kappa(w)i(\kappa(w))^{-1}\underline{\mu}(w)^2 \end{aligned}$$

This implies $\kappa(w) \in k^*$ and hence $w \in \Gamma(A, \sigma)(k)$. Thus α is a spinor norm, being the similarity of an element in the special Clifford group. Also note if α is a spinor norm, then $\alpha = \underline{\mu}(\gamma)$ for some $\gamma \in \Gamma(A, \sigma)(k)$ and $\mu_*(\gamma) = (\underline{\mu}(\gamma), \underline{\mu}(\gamma)^2)$.

Thus a special element (f, z) is in the image of μ_* if and only if the produced scalar α is a spinor norm. We call the class of α in $\frac{k^*}{\text{Sn}(A, \sigma)}$ to be the scalar obstruction preventing the special element $(f, z) \in U(k)$ from being in the image $\mu_*(\Omega(A, \sigma)(k))$.

4.3. An obstruction to being in the image of $\underline{\mu}$ for n even. Given $z \in Z^*$, we would like to formulate an obstruction which prevents z from being in the image $\underline{\mu}(\Omega(A, \sigma)(k))$. Note that for $z \in Z^*$, $\underline{\mu}(z) = z^2$ and hence the subgroup Z^{*2} is in the image $\underline{\mu}(\Omega(A, \sigma)(k))$.

Like in the case of odd n , we can construct the map $S : \text{PGO}^+(A, \sigma)(k) \rightarrow \frac{Z^*}{Z^{*2}}$ induced by the following commutative diagram with exact rows [9, Def. 13.32, p. 187]:

$$\begin{array}{ccccccc} 1 & \longrightarrow & Z^* & \longrightarrow & \Omega(A, \sigma)(k) & \xrightarrow{\chi'} & \text{PGO}^+(A, \sigma)(k) & \longrightarrow & 1 \\ & & \downarrow \underline{\mu} & & \downarrow \underline{\mu} & & \downarrow S & & \\ 1 & \longrightarrow & Z^{*2} & \longrightarrow & Z^* & \longrightarrow & \frac{Z^*}{Z^{*2}} & \longrightarrow & 1 \end{array}$$

Again by the surjectivity of the maps, $\underline{\mu} : Z^* \rightarrow Z^{*2}$ and $\chi' : \Omega(A, \sigma)(k) \rightarrow \text{PGO}^+(A, \sigma)(k)$, an element $z \in Z^*$ is in the image $\underline{\mu}(\Omega(A, \sigma)(k))$ if and only

if its image $[z] \in \frac{Z^*}{Z^{*2}}$ is in the image $S(\text{PGO}^+(A, \sigma)(k))$. Therefore we look for an obstruction preventing $[z]$ from being in the image $S(\text{PGO}^+(A, \sigma)(k))$. And as before, we arrive at the the following commutative diagram (Figure 2) with exact rows and columns [9, Prop. 13.33, p. 188], where

$$\begin{array}{ccc}
 \text{O}^+(A, \sigma)(k) & \xrightarrow{Sn} & \frac{k^*}{k^{*2}} \\
 \downarrow \pi & & \downarrow i \\
 \text{PGO}^+(A, \sigma)(k) & \xrightarrow{S} & \frac{Z^*}{Z^{*2}} \\
 \downarrow \mu & & \downarrow j \\
 \frac{k^*}{k^{*2}} & = & \frac{k^*}{k^{*2}}
 \end{array}$$

Figure 2. Spinor norms and S for n even

$\mu : \text{PGO}^+(A, \sigma)(k) \rightarrow \frac{k^*}{k^{*2}}$ is induced by the multiplier map $\mu : \text{GO}^+(A, \sigma) \rightarrow \mathbb{G}_m$

$i : \frac{k^*}{k^{*2}} \rightarrow \frac{Z^*}{Z^{*2}}$ is the inclusion map

$j : \frac{Z^*}{Z^{*2}} \rightarrow \frac{k^*}{k^{*2}}$ is induced by the norm map from $Z^* \rightarrow k^*$.

Definition 4.2. We call an element $z \in Z^*$ to be *special* if there exists a $[g] \in \text{PGO}^+(A, \sigma)(k)$ such that $j([z]) = \mu([g])$.

Let $z \in Z^*$ be a special element and let $[g] \in \text{PGO}^+(A, \sigma)(k)$ be such that $j([z]) = \mu([g])$. As before a *special* element $z \in Z^*$ is in the image $\underline{\mu}(\Omega(A, \sigma)(k))$ if and only if $[z]$ is in the image $S(\text{PGO}^+(A, \sigma)(k))$.

Thus $S([g])[z]^{-1}$ is in kernel $j = \text{Image } i$ and hence there exists some $\alpha \in k^*$ such that

$$[z] = S([g])[\alpha] \in \frac{Z^*}{Z^{*2}}.$$

Note that if g is changed by an element in $\text{O}^+(A, \sigma)(k)$, then α changes by a spinor norm by Figure 2 above. Thus given a special element, we have produced a scalar $\alpha \in k^*$ which is well defined up to spinor norms.

$$\begin{aligned}
 [z] \in S(\text{PGO}^+(A, \sigma)(k)) & \iff [\alpha] \in S(\text{PGO}^+(A, \sigma)(k)) \\
 & \iff (\alpha) \in \underline{\mu}(\Omega(A, \sigma)(k)).
 \end{aligned}$$

Since $\alpha \in k^*$ also, this is equivalent to α being a spinor norm [9, Prop. 13.25, p. 184].

We call the class of α in $\frac{k^*}{\text{Sn}(A, \sigma)}$ to be the scalar obstruction preventing the *special* element $z \in Z^*$ from being in the image $\underline{\mu}(\Omega(A, \sigma)(k))$.

4.4. Scharlau’s norm principle for $\mu : \text{GO}^+(A, \sigma) \rightarrow \mathbb{G}_m$. Let $\mu : \text{GO}^+(A, \sigma) \rightarrow \mathbb{G}_m$ denote the multiplier map and let L/k be a separable field extension of finite degree. Let $g_1 \in \text{GO}^+(A, \sigma)(L)$ be such that $\mu(g_1) = f_1 \in L^*$. Let f denote $N_{L/k}(f_1)$. We would like to show that f is in the image $\mu(\text{GO}^+(A, \sigma)(k))$.

Note that by a generalization of Scharlau’s norm principle ([9, Prop. 12.21]; [3, Lemma 4.3]) there exists a $\tilde{g} \in \text{GO}(A, \sigma)(k)$ such that $f = \mu(\tilde{g})$. However we would like to find a *proper* similitude $g \in \text{GO}^+(A, \sigma)(k)$ such that $\mu(g) = f$.

We investigate the cases when the algebra A is non-split and split separately.

Case I: A is non-split. Note that $g_1 \in \text{GO}^+(A, \sigma)(L)$. If $\tilde{g} \in \text{GO}^+(A, \sigma)(k)$, we are done. Hence assume $\tilde{g} \notin \text{GO}^+(A, \sigma)(k)$. By a generalization of Dieudonné’s theorem [9, Thm. 13.38, p. 190], we see that the quaternion algebras

$$\begin{aligned} B_1 &= (Z, f_1) = 0 \in \text{Br}(L), \\ B_2 &= (Z, f) = A \in \text{Br}(k). \end{aligned}$$

Since A is non-split, $B_2 \neq 0 \in \text{Br}(k)$. However co-restriction of B_1 from L to k gives a contradiction, because

$$0 = \text{Cor } B_1 = (Z, N_{L/k}(f_1)) = B_2 \in \text{Br}(k).$$

Hence $\tilde{g} \in \text{GO}^+(A, \sigma)(k)$.

Case II: A is split. Since A is split, $A = \text{End } V$ where (V, q) is a quadratic space and σ is the adjoint involution for the quadratic form q . Again, if $\tilde{g} \in \text{GO}^+(A, \sigma)(k)$, we are done. Hence assume $\tilde{g} \notin \text{GO}^+(A, \sigma)(k)$. That is

$$\det(\tilde{g}) = -f^{2n/2} = -(f^n).$$

Since A is of even degree $(2n)$ and split, there exists an isometry¹ h of determinant -1 . Set $g = \tilde{g}h$. Then $\det(g) = f^n$ where $\mu(g) = f$. Thus we have found a suitable $g \in \text{GO}^+(A, \sigma)(k)$ which concludes the proof of the following:

Theorem 4.3. *The norm principle holds for the map $\mu : \text{GO}^+(A, \sigma) \rightarrow \mathbb{G}_m$.*

4.5. Spinor obstruction to norm principle for non-trialitarian D_n . Let L/k be a separable field extension of finite degree. And let $w_1 \in \Omega(A, \sigma)(L)$ be such that for

$$\begin{aligned} n \text{ odd} : \mu_*(w_1) &= \theta \text{ which is equal to } (f_1, z_1) \in U(L), \\ n \text{ even} : \underline{\mu}(w_1) &= \theta \text{ which is equal to } z_1 \in (R_{Z/k}\mathbb{G}_m)(L). \end{aligned}$$

¹Since V is of even dimension $2n$, h can be chosen to be a hyperplane reflection for instance

We would like to investigate whether $N_{L/k}(\theta)$ is in the image of $\mu_* (\Omega (A, \sigma) (k))$ (resp. $\underline{\mu} (\Omega (A, \sigma) (k))$) when n is odd (resp. even) in order to check if the norm principle holds for the map $\mu_* : \Omega (A, \sigma) \rightarrow U$ (resp. $\underline{\mu} : \Omega (A, \sigma) \rightarrow R_{Z/k} \mathbb{G}_m$).

Let $[g_1] \in \text{PGO}^+ (A, \sigma) (L)$ be the image of w_1 under the canonical map $\chi' : \Omega(A, \sigma)(L) \rightarrow \text{PGO}^+ (A, \sigma)(L)$. Clearly θ is *special* and let $g_1 \in \text{GO}^+ (A, \sigma) (L)$ be such that $\mu([g_1]) = j([\theta])$.

By Theorem 4.3, there exists a $g \in \text{GO}^+ (A, \sigma) (k)$ such that²

$$\mu([g]) = N_{L/k} (j[\theta]) = j ([N_{L/k} \theta]) .$$

Hence $N_{L/k}(\theta)$ is *special*.

By Subsection 4.2 (resp. 4.3) , $N_{L/k}(\theta)$ is in the image of μ_* (resp $\underline{\mu}$) if and only if the scalar obstruction $\alpha \in \frac{k^*}{\text{Sn}(A, \sigma)}$ defined for $N_{L/k}(\theta)$ vanishes. Thus we have a spinor norm obstruction given below.

Theorem 4.4 (Spinor norm obstruction). *Let L/k be a finite separable extension of fields. Let f denote the map μ_* (resp $\underline{\mu}$) in the case when n is odd (resp. even). Given $\theta \in f (\Omega (A, \sigma) (L))$, there exists scalar obstruction $\alpha \in k^*$ such that*

$$N_{L/k}(\theta) \in f (\Omega (A, \sigma) (k)) \iff \alpha = 1 \in \frac{k^*}{\text{Sn}(A, \sigma)} .$$

Thus the norm principle for the canonical map

$$\Omega (A, \sigma) \rightarrow \frac{\Omega (A, \sigma)}{[\Omega (A, \sigma), \Omega (A, \sigma)]}$$

and hence for non-trialitarian D_n holds if and only if the scalar obstructions are spinor norms.

5. Quasi-split groups

Let G be a connected reductive k -group whose Dynkin diagram does not contain connected components of type E_8 and let G' denote its derived subgroup. Let G^{sc} denote the simply connected cover of G' . Then one has the exact sequence $1 \rightarrow C \rightarrow G^{sc} \rightarrow G' \rightarrow 1$, where C is a finite k -group of multiplicative type, central in G^{sc} . Assuming that G^{sc} is quasi-split, we would like to show that G satisfies SQ by following the reduction techniques used in Sections 2 and 3.

Lemma 5.1. *Let G be a connected reductive k -group. If G^{sc} is quasi-split, then there exists an extension $1 \rightarrow Q \rightarrow H \xrightarrow{\psi} G \rightarrow 1$, where Q is a quasi-trivial k -torus, central in reductive k -group H with H' simply connected and quasi-split.*

²The map j commutes with $N_{L/k}$ in both cases.

Proof. Recall that there is a central extension (called a z -extension) of G by a quasi-trivial torus Q such that H' is semisimple and simply connected ([11, Prop. 3.1] and [4, Lemma 1.1.4]).

$$1 \rightarrow Q \rightarrow H \xrightarrow{\psi} G \rightarrow 1.$$

The restriction $\psi|_{H'} : H' \rightarrow G$ yields the fact that H' is the simply connected cover of G' and hence is quasi-split. \square

Lemmata 2.2 and 5.1 imply that we can restrict ourselves to connected reductive k -groups G such that G' is simply connected and quasi-split.

Lemma 5.2. *Let H be any reductive k -group such that its derived subgroup H' is semisimple simply connected and quasi-split. Let T denote the k -torus H/H' .*

Then the natural exact sequence $1 \rightarrow H' \rightarrow H \xrightarrow{\phi} T \rightarrow 1$ induces surjective maps $\phi(L) : H(L) \rightarrow T(L)$ for all field extensions L/k . In particular, the norm principle holds for $\phi : H \rightarrow T$.

Proof. There exists a quasi-trivial maximal torus Q_1 of H' defined over k [8, Lem. 6.7]. Let $Q_1 \subset Q_2$, where Q_2 is a maximal torus of H defined over k . The proof of [8, Lem. 6.6] shows that $\phi|_{Q_2} : Q_2 \rightarrow T$ is surjective and that $Q_2 \cap H'$ is a maximal torus of H' . Since $Q_2 \cap H' \subseteq Q_1$, we get the following extension of k -tori

$$1 \rightarrow Q_1 \rightarrow Q_2 \rightarrow T \rightarrow 1$$

Since Q_1 is quasitrivial, $H^1(L, Q_1) = 0$ for any field extension L/k which gives the surjectivity of $\phi(L) : Q_2(L) \rightarrow T(L)$ and hence of $\phi(L) : H(L) \rightarrow T(L)$. \square

Let \hat{G} be an envelope of G' defined using an embedding of $\mu = Z(G')$ into a quasi-trivial torus S . Note that G' is assumed to be simply connected and quasi-split and is also the derived subgroup of \hat{G} by construction.

$$\begin{array}{ccc} \mu & \xrightarrow{\delta} & G' \\ \downarrow \rho & & \downarrow \\ S & \xrightarrow{\gamma} & \hat{G} \end{array}$$

Thus, we get an exact sequence $1 \rightarrow G' \rightarrow \hat{G} \rightarrow \hat{G}/G' \rightarrow 1$ to which we can apply Lemma 5.2 to conclude that the norm principle holds for the canonical map $\hat{G} \rightarrow \frac{\hat{G}}{[\hat{G}, \hat{G}]}$.

Constructing the intermediate group \tilde{G} as in Section 3.1, we see that the norm principle also holds for the natural map $\tilde{G} \rightarrow \tilde{G}/G$ [1, Prop. 5.1]. Then using Theorem 3.1 [3], Lemma 3.2, and a remark from Gopal Prasad that G^{sc} is quasi-split if and only if G is quasi-split, we can conclude that Theorem 1.3 (restated below) holds.

Theorem 1.3. *Let k be a field of characteristic not 2. Let G be a connected quasi-split reductive k -group whose Dynkin diagram does not contain connected components of type E_8 . Then Serre's question has a positive answer for G .*

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