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On N-Numbers

1. *Definition of the N-numbers*

We consider the expansion of $(x + x^{-1})^n$ in function of terms of the form $x^m + x^{-m}$. We clearly have

$$(x + x^{-1})^n = \sum_{m=0}^{n} \binom{n}{m} x^{2m-n}. \tag{1}$$

We separate two possible cases:

(i) n is even, $n = 2p$, then

$$(x + x^{-1})^{2p} = \sum_{m=0}^{2p} \binom{2p}{m} x^{2m-2p} = \sum_{m=0}^{p-1} \binom{2p}{m} [x^{2p-2m} + x^{2m-2p}] + \binom{2p}{p}. \tag{2}$$

(ii) n is odd, $n = 2p + 1$

$$(x + x^{-1})^{2p+1} = \sum_{m=0}^{2p+1} \binom{2p+1}{m} x^{2m-2p-1} = \sum_{m=0}^{p} \binom{2p+1}{m} [x^{2p-2m+1} + x^{2m-2p-1}]. \tag{3}$$

We consider now the inverse problem, i.e. expressing $x^n + x^{-n}$ in function of powers of $x + x^{-1}$. We call N-numbers the coefficients of this expansion. It is immediately seen that if n is even the expansion will contain only even powers of $x + x^{-1}$, while if n is odd the expansion will contain only odd powers of $x + x^{-1}$. We thus write

(i) for n even, $n = 2p$,

$$x^{2p} + x^{-2p} = \sum_{m=0}^{p} N_{2m}^{2p} (x + x^{-1})^{2m}; \quad N_{2m+1}^{2p} = 0, \quad N_{2m}^{2p} = 0 \quad \text{for} \quad p < m, \quad p < 0, \quad m < 0; \tag{4}$$

(ii) for n odd, $n = 2p + 1$,

$$x^{2p+1} + x^{-2p-1} = \sum_{m=0}^{p} N_{2m+1}^{2p+1} (x + x^{-1})^{2m+1}; \quad N_{2m}^{2p+1} = 0, \quad N_{2m+1}^{2p+1} = 0 \quad \text{for} \quad p < m, \quad p < 0, \quad m < 0. \tag{5}$$

It is the aim of this paper to study some of the properties of the N-numbers, in particular their relation to the binomial coefficients.

The N-numbers are known in the literature. We find, for example, in [1], pp. 201–205, an explicit definition of these numbers. Here is how this is done: Let,

$$y^2 - S y + P = 0,$$

be a quadratic equation, α and β its roots, then clearly $\alpha + \beta = S$, $\alpha\beta = P$. According to [1] we have the following expression for $\alpha^n + \beta^n$:

(i) If $n = 2p$, i.e. n is even

$$\alpha^{2p} + \beta^{2p} = 2(-1)^p [P^p + \sum_{m=1}^{p} (-1)^m \frac{p^2 (p-1)^2 \dots [p^2 - (m-1)^2]}{(2m)!} S^{2m} P^{p-m}]. \tag{4'}$$

(ii) If $n = 2p + 1$, i.e. n is odd

$$\alpha^{2p+1} + \beta^{2p+1} = (-1)^p (2p+1) \times$$
$$\left[S P^p + \sum_{m=1}^{p} (-1)^m \frac{(p+m)\, p\, (p^2 - 1^2)(p^2 - 2^2) \dots [p^2 - (m-1)^2]}{(2m+1)!} S^{2m+1} P^{p-m}\right]. \quad (5')$$

Since we are interested in the coefficients of this expansion we can, without loss of generality assume that $P = 1$, i.e., $\beta = 1/\alpha$. Taking $\alpha = x$, $\beta = x^{-1}$, we obtain the initial definition of the N-numbers. All results obtained in the subsequent calculations can be checked on formulae (4') and (5').

The proofs given in [1] are based on the expansion of

$$A = (2 - S y)(1 - S y + P y^2)^{-1},$$

in powers of y. By writing $S = 2x$, $P = 1$, we obtain

$$A = 2(1 - x y)(1 - 2 x y + y^2)^{-1},$$

which according to [2], page 223, is the generating function of the Chebyshev polynomials, i.e.

$$T(x, y) = \frac{A}{2} = (1 - x y)(1 - 2 x y + y^2)^{-1} = \sum_{m=0}^{\infty} T_m(x)\, y^m.$$

It follows that the Chebyshev polynomials can be expressed in function of the N-numbers by the following relations

$$T_{2p}(x) = \frac{1}{2} \sum_{m=0}^{p} N_{2m}^{2p} (2x)^{2m}, \quad T_{2p+1}(x) = \frac{1}{2} \sum_{m=0}^{p} N_{2m+1}^{2p+1} (2x)^{2m+1}.$$

2. *Convolution Relations*

We separate again two cases:

(i) If n is even, $n = 2p$, we obtain from (4) and (2)

$$x^{2p} + x^{-2p} = \sum_{m=0}^{p} N_{2m}^{2p} (x + x^{-1})^{2m} = \sum_{m=0}^{p} N_{2m}^{2p} \left[\sum_{s=0}^{m-1} \binom{2m}{s} (x^{2m-2s} + x^{2s-2m}) + \binom{2m}{m}\right].$$

Let $m - s = q$, hence, $s = m - q \leqslant p - q$, so that,

$$x^{2p} + x^{-2p} = \sum_{q=1}^{p} (x^{2q} + x^{-2q}) \sum_{s=0}^{p-q} \binom{2q+2s}{s} N_{2q+2s}^{2p} + \sum_{s=0}^{p} \binom{2s}{s} N_{2s}^{2p}. \quad (6)$$

From (6) it follows immediately that

$$\sum_{s=0}^{p-q} \binom{2q+2s}{s} N_{2q+2s}^{2p} = \delta_p^q, \quad q = 1, 2, \dots, p, \quad (7)$$

which is a convolution relation between the N-numbers and the binomial coefficients, for n even.

(ii) If n is odd, $n = 2p + 1$, we obtain from (5) and (3)

$$x^{2p+1} + x^{-2p-1} = \sum_{m=0}^{p} N_{2m+1}^{2p+1} (x + x^{-1})^{2m+1}$$
$$= \sum_{m=0}^{m} N_{2m+1}^{2p+1} \left[\sum_{s=0}^{m} \binom{2m+1}{s} (x^{2m-2s+1} + x^{2s-2m-1}) \right].$$

Let $m - s = q$, hence, $s = m - q \leqq p - q$, so that,

$$x^{2p+1} + x^{-2p-1} = \sum_{q=0}^{p} (x^{2q+1} + x^{-2q-1}) \sum_{s=0}^{p-q} \binom{2q+2s+1}{s} N_{2q+2s+1}^{2p+1}. \tag{8}$$

From (8) it follows immediately that

$$\sum_{s=0}^{p-q} \binom{2q+2s+1}{s} N_{2q+2s+1}^{2p+1} = \delta_p^q, \tag{9}$$

which is a convolutional relation between the N-numbers and the binomial coefficients, for n odd.

3. *Recurrence Relations*

Separating the even and odd cases we have
(i) for n even, $n = 2p$, from (4)

$$(x + x^{-1})^2 (x^{2p} + x^{-2p}) = \sum_{m=0}^{p} N_{2m}^{2p} (x + x^{-1})^{2m+2},$$

or,

$$(x^{2p+2} + x^{-2p-2}) + (x^{2p-2} + x^{-2p+2}) + 2(x^{2p} + x^{-2p}) = \sum_{m=0}^{p} N_{2m}^{2p} (x + x^{-1})^{2m+2}.$$

Using (4) we obtain

$$\sum_{m=0}^{p+1} N_{2m}^{2p+2} (x + x^{-1})^{2m} + \sum_{m=0}^{p-1} N_{2m}^{2p-2} (x + x^{-1})^{2m} + 2 \sum_{m=0}^{p} N_{2m}^{2p} (x + x^{-1})^{2m} =$$
$$= \sum_{m=0}^{p} N_{2m}^{2p} (x + x^{-1})^{2m+2},$$

so that equating the coefficients of $(x + x^{-1})^{2m}$ we obtain

$$N_{2m}^{2p+2} + N_{2m}^{2p-2} + 2 N_{2m}^{2p} = N_{2m-2}^{2p},$$

or, changing p into $p - 1$, we obtain

$$N_{2m}^{2p} = N_{2m-2}^{2p-2} - N_{2m}^{2p-4} - 2 N_{2m}^{2p-2}. \tag{10}$$

(ii) For n odd, i.e. $n = 2p + 1$, we have from (5),

$$(x + x^{-1})^2 (x^{2p+1} + x^{-2p-1}) = \sum_{m=0}^{p} N_{2m+1}^{2p+1} (x + x^{-1})^{2m+3}.$$

Operating as in (i) we obtain

$$N^{2p+1}_{2m+1} = N^{2p-1}_{2m-1} - N^{2p-3}_{2m+1} - 2\,N^{2p-1}_{2m+1}. \tag{11}$$

It will however be seen in the next section that this relation holds only for $1 < p$. Both (10) and (11) can be written

$$N^{n}_{q} = N^{n-2}_{q-2} - N^{n-4}_{q} - 2\,N^{n-2}_{q}, \tag{12}$$

where, for $n = 2\,p$, $q = 2\,m$, and, for $n = 2\,p + 1$, $q = 2\,m + 1$, $p > 1$.

4. *Numerical Results*

(i) For n even, $n = 2\,p$, we take $N^0_0 = 1$, by definition. We then obtain from (10), observing that $N^{2p-2}_{-2} = 0$,

$$N^{2p}_{0} = -N^{2p-4}_{0} - 2\,N^{2p-2}_{0},$$

thus, $N^2_0 = -2$, $N^4_0 = +2, \cdots$, thus assuming that $N^{2p-4}_0 = (-1)^p\,2$, and, $N^{2p-2}_0 = (-1)^{p+1}\,2$, we obtain, $N^{2p}_0 = -(-1)^p\,2 - 2\,(-1)^{p+1}\,2 = (-1)^p\,2$, thus

$$N^{2p}_{0} = (-1)^p\,2. \tag{13}$$

Again, starting from (10) we have

$$N^{2p}_{2} = N^{2p-2}_{0} - N^{2p-4}_{2} - 2\,N^{2p-2}_{2},$$

thus, since for negative indices the N-numbers are zero, we have

$$N^2_2 = N^0_0 - N^{-2}_2 - 2\,N^0_2 = N^0_0 = 1$$

$$N^4_2 = N^2_0 - N^0_2 - 2\,N^2_2 = -2 - 2\cdot 1 = -4$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Since, $N^{2p-2}_0 = (-1)^{p+1}\,2$, according to (13), assuming

$$N^{2p-4}_{2} = (-1)^{p+1}\,(p-2)^2, \quad N^{2p-2}_{2} = (-1)^{p}\,(p-1)^2,$$

we obtain,

$$N^{2p}_{2} = (-1)^{p+1}\,2 - (-1)^{p+1}\,(p-2)^2 - 2\,(-1)^p\,(p-1)^2$$

so that

$$N^{2p}_{2} = (-1)^{p+1}\,p^2. \tag{14}$$

Again using (10) we have, using the conditions of (4),

$$N^{2p}_{2p} = N^{2p-2}_{2p-2} - N^{2p-4}_{2p} - 2\,N^{2p-2}_{2p} = N^{2p-2}_{2p-2} = N^0_0 = 1,$$

thus

$$N^{2p}_{2p} = 1, \tag{15}$$

and,

$$N^{2p}_{2p-2} = N^{2p-2}_{2p-4} - N^{2p-4}_{2p-2} - 2\,N^{2p-2}_{2p-2} = N^{2p-2}_{2p-4} - 2\cdot 1\,,$$

thus assuming, $N^{2p-2}_{2p-4} = -(2\,p-2)$, we obtain

$$N^{2p}_{2p-2} = -2\,p\,. \tag{16}$$

With the preceding information and (10) we can construct a table of N^{2p}_{2m}:

	$m =$	0	1	2	3	4
	$2\,m =$	0	2	4	6	8
$p = 0$	$2\,p = 0$	1				
1	2	-2	1			
2	4	2	-4	1		
3	6	-2	9	-6	1	
4	8	2	-16	20	-8	1.

(ii) n is odd, $n = 2\,p + 1$. By definition we clearly have $N^1_1 = 1$. Since $x^3 + x^{-3} = -3\,(x + x^{-1}) + (x + x^{-1})^3$, thus $N^3_1 = -3$, $N^3_3 = 1$. But according to (11), and the conditions of (5)

$$N^3_1 = N^1_{-1} - N^{-1}_1 - 2\,N^1_1 = -2\,N^1_1 = -2\,,$$

which is clearly wrong. This is why we add the condition $p > 1$. Continuing the same way we have

$$N^5_1 = -N^1_1 - 2\,N^3_1 = -1 - 2\,(-3) = 5\,,$$

more generally assuming that
$N^{2p-3}_1 = (-1)^p\,(2\,p-3)$, and, $N^{2p-1}_1 = (-1)^{p+1}\,(2\,p-1)$, we obtain,

$$N^{2p+1}_1 = -(-1)^p\,(2\,p-3) - 2\,(-1)^{p+1}\,(2\,p-1) = (-1)^p\,(2\,p+1)\,,$$

so that

$$N^{2p+1}_1 = (-1)^p\,(2\,p+1)\,, \quad p > 1\,. \tag{17}$$

Similarly,

$$N^{2p+1}_{2p+1} = N^{2p-1}_{2p-1} - N^{2p-3}_{2p+1} - 2\,N^{2p-1}_{2p} = N^{2p-1}_{2p-1} = N^1_1 = 1\,,$$

thus

$$N^{2p+1}_{2p+1} = 1\,. \tag{18}$$

Similarly,

$$N^{2p+1}_{2p-1} = N^{2p-1}_{2p-3} - N^{2p-3}_{2p-1} - 2\,N^{2p-1}_{2p-1} = N^{2p-1}_{2p-3} - 2\,N^{2p-1}_{2p-1}\,,$$

thus, since $N^{2p-1}_{2p-1} = 1$, according to (18), we have $N^3_1 = -3$, $N^5_3 = -5\,,\cdots$, so that, $N^{2p+1}_{2p-1} = -(2\,p+1)\,.$

With these informations and (11) we can construct the following table of values of N_{2m+1}^{2p+1}:

	$m =$	0	1	2	3	4
	$2m+1 =$	1	3	5	7	9
$p = 0$	$2p+1 = 1$	1				
1	3	-3	1			
2	5	5	-5	1		
3	7	-7	14	-7	1	
4	9	9	-30	27	-9	1

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REFERENCES

[1] B. Chrystal, *Algebra* (Chelsea Publ. Co., N.Y., 1952).
[2] J. Riordan, *An Introduction to Combinatorial Analysis* (J. Wiley Inc., N.Y., 1958).

Kleine Mitteilungen

Über die Kompatibilität gewisser Ebenenabbildungen und linearer Punktabbildungen

Wir betrachten eine eineindeutige Abbildung zwischen den Elementen zweier Ebenenbündel der n-dimensionalen affinen Räume E_n und E_n'. Es entsteht die Frage, ob zwischen E_n und E_n' eine solche eineindeutige Punktabbildung existiert, welche die gegebene Ebenenabbildung induziert, das heisst, die mit der obigen Ebenenabbildung kompatibel ist. – Das Ziel dieser Note ist, bezüglich dieser Frage den folgenden Satz zu beweisen:

Satz: Eine eineindeutige Ebenenabbildung zwischen den Elementen zweier Ebenenbündel ist dann und nur dann mit einer eineindeutigen Punktabbildung kompatibel, wenn diese Ebenenabbildung ein Ebenenbüschel in beiden Richtungen wieder in Ebenenbüschel überführt. Die Ebenenabbildung bestimmt die Punktabbildung bis auf eine Dehnung.

Die Notwendigkeit der Bedingungen ist offenbar.

Wir zeigen, dass die Bedingung auch hinreichend ist. Für die Träger der Ebenenbündel wollen wir der Einfachheit halber den Ursprung O bzw. O' von E_n bzw. E_n' wählen. Die Zuordnung der Träger der einander entsprechenden Ebenenbüschel induziert zwischen den Strahlenbündeln mit den Trägern O bzw. O' eine eineindeutige Strahlenabbildung. Diese Strahlenabbildung ordnet drei koplanaren Strahlen drei Strahlen mit derselben Eigenschaft zu. Sind nämlich s, t und u drei verschiedene koplanare Strahlen, so gehört ihre Ebene σ zu jedem der Ebenenbüschel mit den Trägern s, t und u. Daher gehört aber die σ nach der Ebenenabbildung zugeordnete Bildebene σ' zu jedem der drei Ebenbüschel mit den Trägern s', t' und u', wo s', t', u' die Bilder nach der Strahlenabbildung von s, t und u sind.

Es sei nun A_{n-1} ein $(n-1)$-dimensionaler Unterraum von E_n, der nicht durch O geht. Mit Hilfe der Strahlenabbildung kann man dem $(n-1)$-dimensionalen Unterraum, der zu A_{n-1} parallel ist und durch O geht, in E_n' eindeutig einen durch O' gehenden $(n-1)$-dimen-