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Autor: Mordell, L.J.
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On Some Ternary Quartic Diophantine Equations

There are not known many instances of Diophantine equations

$$f(x, y, z) = 0 \quad (1)$$

representing a non-degenerate quartic surface for which an infinity of integer solutions exist. It may therefore be of interest to give a few.

Theorem 1

The equation

$$z^2 = U_1^2 + U_2 U_3, \quad (2)$$

where

$$U_r = a_r x^2 + h_r x y + b_r y^2 + f_r y + g_r x \quad (r = 1, 2, 3),$$

and the coefficients are integers, has an infinity of integer solutions if for either $r = 2$ or 3 , $h_r^2 - 4 a_r b_r > 0$, and is not a perfect square and U_2 or U_3 is absolutely irreducible.

From (2), we have

$$z + U_1 = \frac{p}{q} U_2, \quad z - U_1 = \frac{q}{p} U_3,$$

where p, q are integers and $(p, q) = 1$. Then

$$2 U_1 = \frac{p}{q} U_2 - \frac{q}{p} U_3. \quad (3)$$

For integer solutions of (3), $U_2 \equiv 0 \pmod{q}$, $U_3 \equiv 0 \pmod{p}$, and then z is also an integer.

Write (3) as

$$P(x, y) = a x^2 + h x y + b y^2 + f y + g x = 0, \quad (4)$$

where

$$a = p^2 a_2 - 2 p q a_1 - q^2 a_3, \quad h = p^2 h_2 - 2 p q h_1 - q^2 h_3,$$

$$b = p^2 b_2 - 2 p q b_1 - q^2 b_3, \quad f = p^2 f_2 - 2 p q f_1 - q^2 f_3,$$

$$g = p^2 g_2 - 2 p q g_1 - q^2 g_3.$$

The equation (4) has a solution $x = 0, y = 0$. GAUSS has shown from a Pellian equation that (4) will have an infinity of integer solutions if $P(x, y)$ is algebraically irreducible and $h^2 - 4 a b > 0$ and is not a perfect square. The condition for reducibility is

$$\Delta = \frac{1}{2} \begin{vmatrix} 2a & h & g \\ h & 2b & f \\ g & f & 2c \end{vmatrix} = 0.$$

This is a binary sextic in p, q and is not identically zero since the coefficient of p^6 is obtained by replacing a, b , etc., in Δ by a_2, b_2 , etc. Hence there will be only a finite

number of values of p and q , if either U_2 or U_3 is irreducible, for which $P(x, y)$ is reducible.

Next

$$h^2 - 4ab = (p^2 h_2 - 2pqh_1 - q^2 h_3)^2 - 4(p^2 a_2 - 2pqa_1 - q^2 a_3)(p^2 b_2 - pqb_1 - q^2 b_3)$$

If $h_2^2 - 4a_2 b_2 > 0$ and is not a perfect square, this holds for $h^2 - 4ab$ if p is large compared with q , and for an infinity of p . This proves Theorem (1).

There are many special cases not included in the theorem. We need only mention

Theorem 2

The equation

$$z^2 = k^2 + x^2(a x^2 + b y^2), \quad a b k \neq 0,$$

has an infinity of integer solutions if k, a, b are integers and either $b > 0$, or $b < 0, 4ak^2 > b^2$.

We have

$$z + k = \frac{q}{p}(a x^2 + b y^2), \quad z - k = \frac{p}{q} x^2,$$

where p, q are integers and $(p, q) = 1$. Then

$$(a q^2 - p^2) x^2 + b q^2 y^2 = 2 k p q.$$

This will have the solution $x = 0, y = t$, where t is an arbitrary integer, if $b q t^2 = 2 k p$, and so if $\delta = (b, 2k)$, we can take

$$\lambda p = \frac{b}{\delta} t^2, \quad \lambda q = \frac{2k}{\delta}, \quad \lambda = \left(t^2, \frac{2k}{\delta} \right)$$

Hence there will be an infinity of integer solutions for x, y if $b(p^2 - a q^2) > 0$ and is not a perfect square, i.e. $b(b^2 t^4 - 4 a k^2) > 0$ and is not a perfect square. This is possible if $b > 0$ for an infinity of values of t , and also if $b < 0, 4 a k^2 > b^2$ for $t = 1$.

The case $4 a k^2 < b^2$ seems difficult. Of course if $a < 0, b < 0$, there are only a finite number of solutions.

L. J. MORDELL, St. Johns College, Cambridge, England

Ungelöste Probleme

Bemerkung zu Nr. 14 (El. Math. 11, 134–135 (1956)). A. a. O. wurde gezeigt, dass die Gleichung $x_1 + x_2 + \dots + x_s = x_1 x_2 \dots x_s$ für jedes natürliche s mindestens eine Lösung in natürlichen Zahlen besitzt. Nach einer Mitteilung von Herrn A. SCHINZEL (Warschau) hat M. MISIUREWICZ vor kurzem bewiesen, dass $s = 2, 3, 4, 6, 24, 144, 174, 444$ die einzigen $s \leq 1000$ sind, für die genau eine Lösung existiert.