Zeitschrift: Elemente der Mathematik

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 22 (1967)

Heft: 5

Rubrik: Kleine Mitteilungen

Jetzt bildet man das Polynom

$$P^*(i\,\omega) = U(\omega^2) + i\,\omega\, V^*(\omega^2) = U^*[-(i\,\omega)^2] + (i\,\omega)\, V^*[-(i\,\omega)^2]\,. \tag{7}$$

Nun setzen wir $i\,\omega = z$ und erhalten aus (7) ein Polynom $P^*(z)$, wobei $P^*(z)$ den Grad n besitzt. Nach unserer Konstruktionsvorschrift ist $P^*(z)$ stabil und $Q^*(z, c)$, das wie in (3) aus $P^*(z)$ gewonnen wird, ist für keine Wahl von c stabil. Somit ist unsere Behauptung bewiesen.

R. Z. DOMIATY, Graz

LITERATUR

[1] W. KAPLAN, *Operational Methods for Linear Systems*, Reading 1962.
[2] A. HURWITZ, *Mathematische Werke*, Band II, Basel 1933.
[3] K. KLOTTER, *Technische Schwingungslehre*, Band 2, Springer-Verlag, Berlin, Göttingen, Heidelberg 1960.
[4] R. ZURMÜHL, *Praktische Mathematik*, Springer-Verlag, Berlin, Heidelberg, New York 1965.

Kleine Mitteilungen

Note on Integration by Residues

In a recent note [3] MELAMED and KAUFMAN point out the advantage of evaluating an integral of the form

$$\int_0^\infty t^\alpha f(t^n)\, dt \tag{1}$$

by considering the integral

$$\int_C z^\alpha f(z^n)\, dz$$

where C is the boundary of an appropriate sector. I should like to suggest a further simplification: begin by making the change of variable $t^n = x$, so that (1) becomes

$$n^{-1} \int_0^\infty x^{(\alpha+1-n)/n} f(x)\, dx\,,$$

which can be evaluated by integrating around the conventional contour for multiple-valued functions (a large circle with a cut along the positive real axis). If we are going to have a multiple-valued function to deal with anyway we may as well keep the rest of the integrand as simple as possible. (Cf. [1], p. 248, problem 83.) Indeed, the change of variable works equally well for the integrals (1) with $\alpha = 0$ considered in [2]; although it introduces a multiple-valued function where none appeared originally, the required residue is often simpler to calculate after the change of variable.

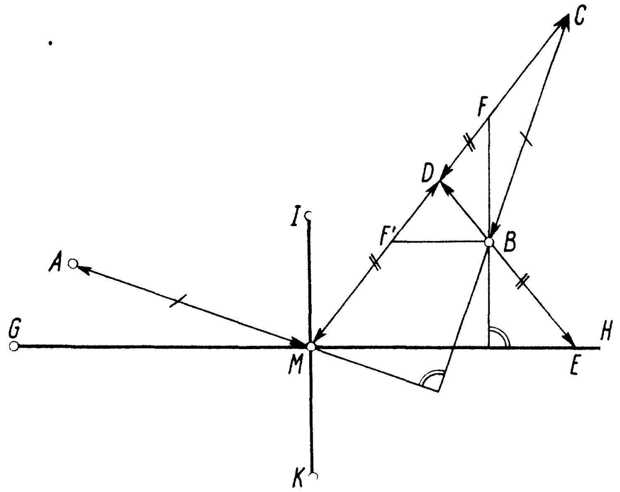
R. P. BOAS, JR., Northwestern University, Evanston, Illinois

REFERENCES

[1] P. FRANKLIN, *Methods of Advanced Calculus* (McGraw-Hill, 1944).
[2] S. MELAMED and H. KAUFMAN, *Evaluation of Certain Improper Integrals by Residues*, Amer. Math. Monthly *72*, 1111–1112 (1965).
[3] S. MELAMED and H. KAUFMAN, *Integration of Multiple-Valued Functions by Residues*, El. Math. *21*, 37–39 (1966).

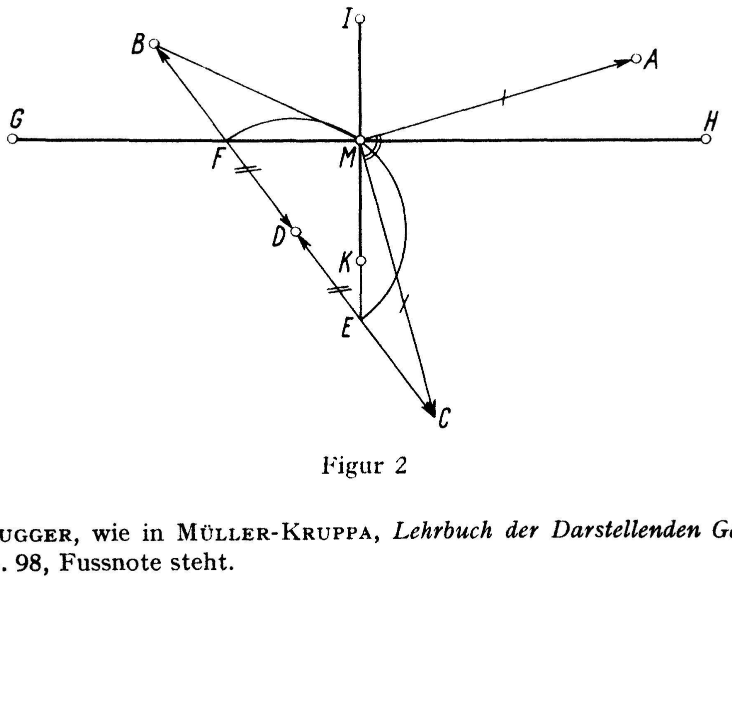
Zur Rytzschen Achsenkonstruktion

Aus dem Nachlass von F. LÖBELL wurde dem Institut für Geometrie an der Technischen Hochschule München eine sehr sorgfältige Niederschrift einer Vorlesung von F. SCHUR über Darstellende Geometrie überlassen. Darin ist die bekannte Konstruktion der Hauptachsen einer Ellipse aus zwei konjugierten Halbmessern nicht wie üblich als «Rytzsche Konstruktion» bezeichnet, sondern es findet sich dort der Vermerk: «FRÉZIER, Coupe des pierres et des bois, 2e éd., t. 1, 1754, p. 159». In der Bayerischen Staatsbibliothek fand sich eine spätere Auflage dieses Werkes und darin die Konstruktion Fig. 1 (gegeben M, A, B; $BC \perp AM$, $BC = AM$, $MD = DC$, $DE = DM$, $BF \perp ME$, $BF' \parallel ME$, $MF = MG = MH = a$, $MF' = BE = MI = MK = b$), also die übliche Konstruktion der Halbachsen a, b mit geringfügigen Abweichungen.



Figur 1

A. PRINGSHEIM, der sich auch mit Fragen zur Geschichte der Mathematik befasste, hat in einem Vortrag bemerkt: «Der Taylorsche Lehrsatz heisst Taylorscher Lehrsatz, trotzdem er von Taylor stammt». Es fragt sich, ob dies vielleicht auch für die Rytzsche Konstruktion zutrifft. Ein Hinweis findet sich in E. MÜLLER, Lehrbuch der Darstellenden Geometrie, 1. Bd., 3. A., Leipzig, Berlin 1920, S. 165, Fussnote auf L. MOSSBRUGGER[1]), Grösstenteils neue Aufgaben aus dem Gebiete der Géometrie descriptive, Zürich 1845. Dort findet sich auf S. 123 u.f. die in Figur 2 wiedergegebene Konstruktion (gegeben



Figur 2

[1]) Nicht MOOSBRUGGER, wie in MÜLLER-KRUPPA, *Lehrbuch der Darstellenden Geometrie*, 4. A., 1936, Leipzig und Berlin, S. 98, Fussnote steht.

$M, A, B; MC \perp MA, MC = MA, BD = DC, BE = MG = MH = a, BF = MI = MK = b$), die in naheliegender Weise abgewandelt, die übliche Konstruktion ergibt und dazu auf S. 125 die Fussnote: «Die oben angegebene Construktion der Achsen der Ellipse aus zwei gegebenen Durchmessern hat mir Hr. RYTZ, Professor der Mathematik an der Gewerbsschule zu Aarau, mitgetheilt; die analytische Herleitung ihrer Richtigkeit habe ich dazugemacht. Der Verfasser». Die Herleitung ist etwas umständlich; im vorangehenden Text schreibt Mossbrugger statt «gegebenen Durchmessern» richtig «zugeordneten Durchmessern».

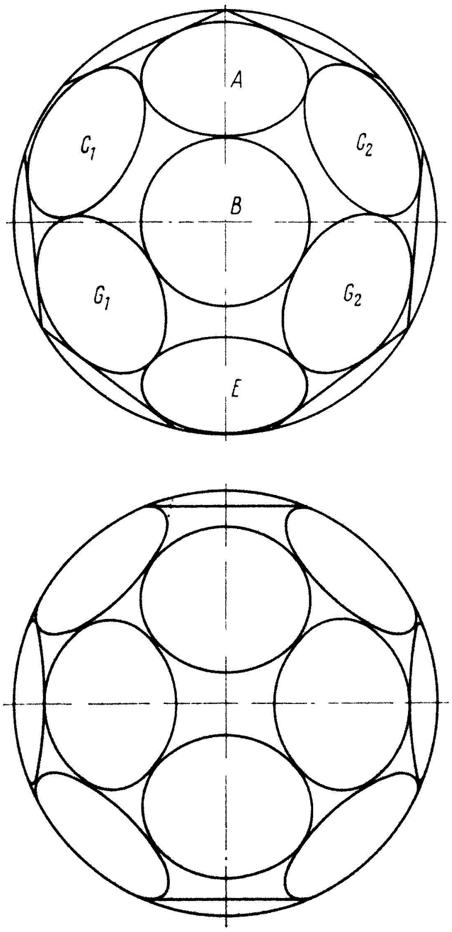
Die bislang nach RYTZ benannte Konstruktion wird daher besser nach FRÉZIER benannt, wenigstens solange, als hierfür kein früherer Autor nachgewiesen ist.

O. BAIER, München

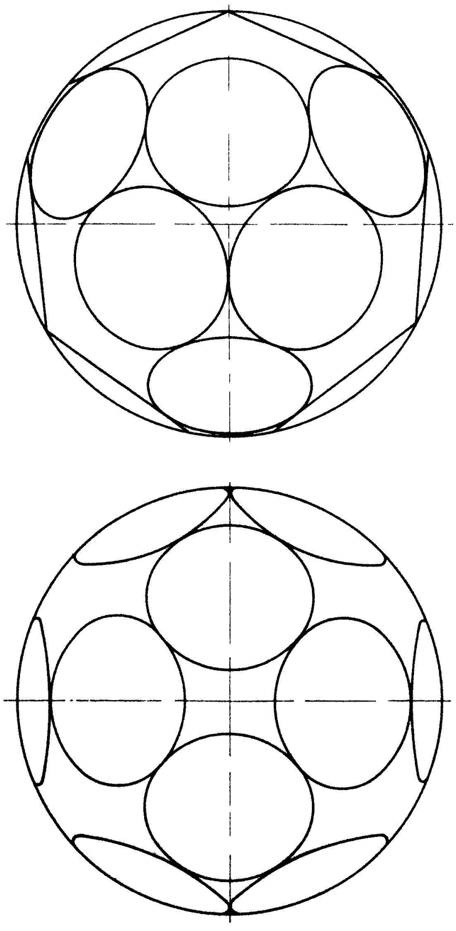
Packing of 19 Equal Circles on a Sphere

Let a_n be the largest possible angular diameter of n equal circles (or spherical caps) which can be packed on the surface of a sphere without overlapping. The value of a_n has been determined [1, 2, 3] for some values of n. For other values of n, conjectured diameters have been computed, based on likely arrangements of the circles, but have not yet been proved to be the best. Summaries of the results are given by FEJES TÓTH [4], COXETER [5], STROHMAJER [6] and GOLDBERG [7]. Conjectured arrangements have been computed for most of the values of n up to $n = 42$. The smallest value of n for which no arrangement has been computed is 19. Suggested values for $n = 18$ and $n = 20$ have been published [6, 8, 9]. These serve as bounds and indications for the possible value to be derived for 19 circles.

For $n = 20$, and the arrangement 20 {1, 3, 3, (6), 3, 3, 1} shown in Figure 1, the published value for a_n is 47°26′. For $n = 18$, and the arrangement 18 {3, 3, (6), 3, 3}



Figur 1, 20 {1,3,3, (6), 3, 3, 1}



Figur 2, 18 {3, 3, (6), 3, 3}

shown in Figure 2, the value of a_n is also $47°26'$. This would seem to indicate that for $n = 19$, the arrangement $19\,\{1, 3, 3, (6), 3, 3\}$ shown in Figure 3 would give the same value. However, a more careful calculation of these values yields $a_n = 47°24'51''$ for $20\,\{1, 3, 3, (6), 3, 3, 1\}$, $a_n = 47°25'52''$ for $18\,\{3, 3, (6), 3, 3\}$ and $a_n = 47°25'22''$ for $19\,\{1, 3, 3, (6), 3, 3\}$.

The value $a = 47°25'22''$ is verified by the following computation. If the polar distance $PA = 2\,u$, then u is obtained from the following equation:

$$\tan u = \cos 60° \tan 47°25'22'', \text{ giving } u = 28°33'15''.$$

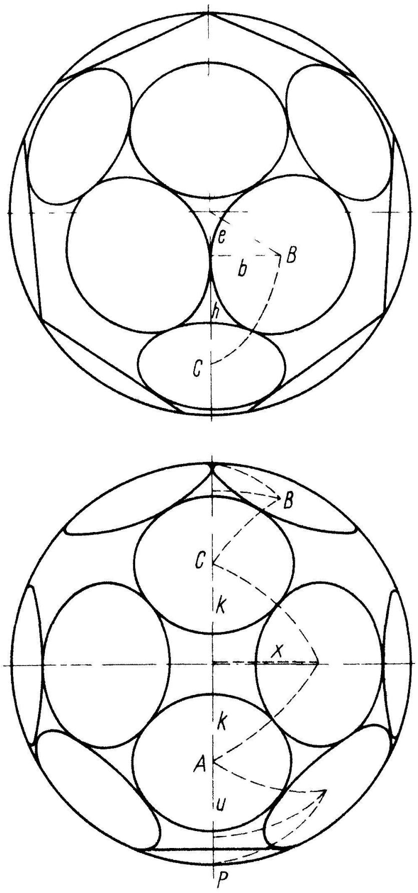
Let $x = 60° - a/2 = 36°17'19''$. If the co-latitude of C is $e + h$, and the distance $AC = 2\,k$, then

$$\cos k = \cos a/\cos x,\ k = 32°55'29'',\ \cos h = \cos a/(\cos a/2),\ h = 42°21'24'',$$

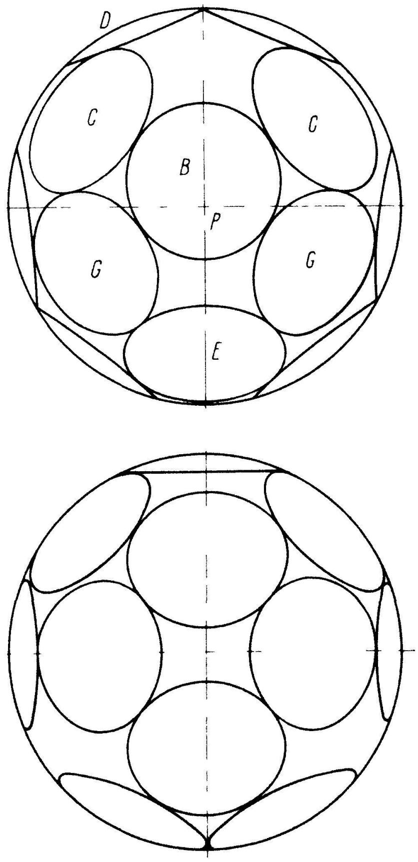
$$\sin b = (\sin a/2)/\sin a,\ b = 33°06'00'',\ \tan e = (\sin a/2)\tan b,\ e = 14°41'21''$$

and $2\,u + 2\,k + h + e = 180°$.

In the arrangement $20\,\{1, 3, 3, (6), 3, 3, 1\}$ shown in Figure 1, the circles have a diameter of $47°24'51''$. Let the circle A be omitted. Increase the diameter of the remaining 19 circles to $47°25'22''$. First, place the circles in the lower hemisphere of Figure 4 in the same manner as in Figure 3. Then, in succession, place the circles E, G and C as shown in the shifted positions of Figure 4. The circle B will move, while remaining in contact with



Figur 3, 19 {1, 3, 3, (6), 3, 3}



Figur 4, 19 {1, 3, 3, (6), ...}

the circle G, until it touches the circles C. However, a computation has shown that, for the selected diameter of these circles, there will be an overlap. Hence, a smaller diameter is required if an overlap is to be avoided for this arrangement.

A summary of the results of the various arrangements of 18, 19 and 20 circles is shown in the following table:

Number and arrangement of circles	Diameter of circles
18 {1, 4, 4, 4, 4, 1}	49° 33′
18 {1, 3, 3, (6), (4), 1}	48° 38′
18 {3, 3, (6), 3, 3}	47° 25′ 52″
19 {1, 3, 3, (6), 3, 3}	47° 25′ 22″
19 {1, 3, 3, (6), ...}	Less than 47° 25′ 22″
20 {1, 3, 3, (6), 3, 3, 1}	47° 24′ 51″

The next gap in the computed set of values of a_n occurs at $n = 23$. No likely arrangements have been suggested. For $n = 24$, a vey efficient arrangement has been determined and it has been proved to be the best possible [3]. If any circle of this arrangement is removed, the remaining 23 circles are locked in stable equilibrium. It is highly probable that this is the best arrangement for 23 circles. For $n = 5$ and $n = 11$, it has already been shown that the best arrangements are obtained by removing a circle from the highly efficient arrangements for $n = 6$ and $n = 12$.

MICHAEL GOLDBERG, Washington, D.C., USA

BIBLIOGRAPHY

[1] SCHÜTTE, K., VAN DER WAERDEN, B. L., *Auf welcher Kugel haben 5, 6, 7, 8 oder 9 Punkte mit Mindestabstand Eins Platz?*, Math. Ann. *123*, 96–124 (1951).
[2] DANZER, L., Proofs for $n = 10$ and 11 have been derived but not yet published.
[3] ROBINSON, R. M., *Arrangement of 24 points on a sphere*, Math. Ann. *144*, 17–48 (1961).
[4] FEJES TÓTH, L., *Lagerungen in der Ebene, auf der Kugel und im Raum* (Berlin, Göttingen, Heidelberg 1953).
[5] COXETER, H. S. M., *The Problem of the Packing of Equal Nonoverlapping Circles on a Sphere*, Trans. New York Academy of Sciences, Ser. II, *24*, 320–331 (1962).
[6] STROHMAJER, J., *Über die Verteilung von Punkten auf der Kugel*, Ann. Univ. Sci. Budapestinensis, Sect. Math. *6*, 49–53 (1963).
[7] GOLDBERG, M., *Axially Symmetric Packing of Equal Circles on a Sphere*, Ann. Univ. Sci. Budapestinensis (to appear).
[8] VAN DER WAERDEN, B. L., *Punkte auf der Kugel*, Drei Zusätze, Math. Ann. *125*, 213–222 (1952).
[9] GOLDBERG, M., *Packing of 18 Equal Circles on the Sphere*, El. Math. *20*, 59–61 (1965).

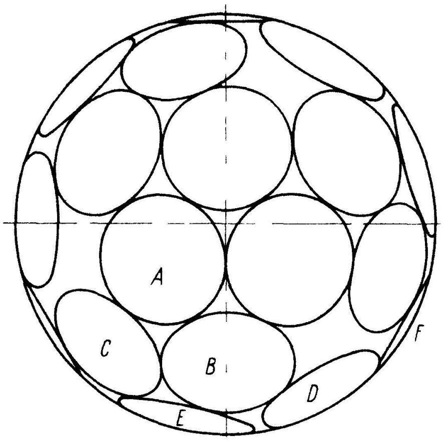
An Improved Packing of 33 Equal Circles on a Sphere

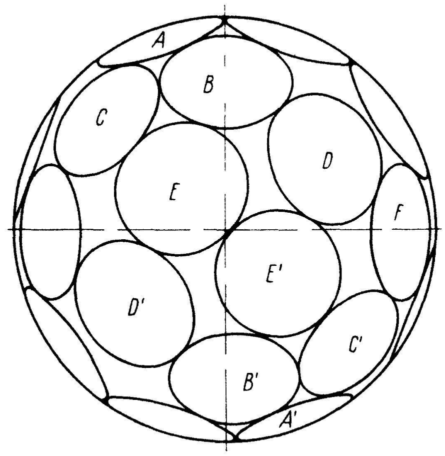
In a previous note [1], the author considered the largest possible angular diameter a_{33} of 33 equal circles (or spherical caps) which can be packed on the surface of a sphere without overlapping. An earlier paper by JUCOVIČ [2] gave this value as 34° 47′. The author exhibited two new arrangements, shown in Figures 1 and 2 of his paper, which yielded larger values, and remarked that both of these arrangements were stable.

However, in a private communication from Professor L. DANZER, it was noted that the better of these two arrangements, shown in Figure 2, was not stable. Hence, there exists a modified arrangement which yields a still larger value. It is the purpose of this note to describe such a new arrangement and to report on the computation of this new value of a_{33} and the locations of the 33 circles on the sphere.

The arrangements are symbolized by a set of numbers whose sum is 33; each number indicates the number of circles on the same latitude. They are equally spaced in longitude except in those cases in which the number is enclosed in a parenthesis.

The new arrangement is shown in the plan and elevation views of the Figure. It consists of eleven bands in each of which three circles are equally distributed in longitude. It has symmetry about the polar axis, and skew symmetry with respect to the equator. The circles in a band do not always lie half-way between the circles of adjacent bands.





The symbol for this arrangement is 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3. By successive approximation, the computed value of a_{33} obtained for this arrangement is $36°15'32''$. In the following table, the latitudes and longitudes of one circle in each band are given. From these, one can verify by computation that adjacent circles do not overlap. For example, if BE is the angular distance between the centers of circles B and E, then

$$\begin{aligned}\cos BE &= \sin 47°08'57'' \ \sin 12°29'52'' + \cos 47°08'57'' \ \cos 12°29'52'' \ \cos 12°42'40'' \\ &= 0{,}73312677 \cdot 0{,}21640175 + 0{,}68009201 \cdot 0{,}97630440 \cdot 0{,}97549189 \\ &= 0{,}80635392 = \cos 36°15'32''.\end{aligned}$$

Hence, circles B and E just touch each other.

Location of Centers of Circles on Sphere

Circle	Latitude	Longitude
A	68°56′34″ N	60°00′00″ W
B	47°08′57″ N	0°
C	33°40′33″ N	45°03′10″ W
D	20°08′07″ N	29°49′13″ E
E	12°29′52″ N	12°42′40″ W
F	0°	60°37′59″ E
E′	12°29′52″ S	13°45′58″ E
D′	20°08′07″ S	18°01′31″ E
C′	33°40′33″ S	47°09′46″ E
B′	47°08′57″ S	2°06′36″ E
A′	68°56′34″ S	62°06′36″ E

The following table shows the comparative values for all the computed arrangements. The quantity D_n, which represents the density of packing on the sphere, is given by $D_n = n\,(1 - \cos 0.5\, a_n)/2$. If the circles have unit diameter, the sphere has the radius $R_n = 1/\sqrt{2 - 2\cos a_n}$.

Arrangement	a	D	R	
3, 9, 9, 9, 3 (unstable)	34°47′	0.755	1.673	JUCOVIČ (1959)
3, 3, (6), (9), (6), 3, 3 (stable)	35°22′	0.780	1.647	GOLDBERG (1963)
3, 3, (6), (9), (6), 3, 3 (unstable)	35°25′	0.782	1.644	GOLDBERG (1963)
3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3 (stable)	36°15′32″	0.819	1.607	GOLDBERG (1966)

Similar improvements were made in the packing of other sets of equal circles on the sphere. They are described in another paper [3].

MICHAEL GOLDBERG, Washington, D.C., USA

BIBLIOGRAPHY

[1] M. GOLDBERG, *Packing of 33 Equal Circles on a Sphere*, El. Math. *18*, 99–100 (1963).

[2] E. JUCOVIČ, *Lagerung von 17, 25 und 33 Punkten auf der Kugel* (Slovak; Russian and German Summaries), Mat.-Fyz. Časopis. Slovensk Akad. Vied. *9*, 173–176 (1959); Math. Rev. *23*, 533 (1962).

[3] M. GOLDBERG, *Axially Symmetric Packing of Equal Circles on a Sphere*, Ann. Univ. Sci. Budapestinensis (to appear).

Aufgaben

Aufgabe 533. Mit A, B, C seien Kreise oder Geraden und auch die Inversion (bzw. Spiegelung) an denselben bezeichnet. Welche geometrische Bedingung erfüllen drei Kreise, wenn in der Möbiusgruppe der Kreistransformationen

$$ABCABCBCABCACBACBCBACB = 1$$

gilt?

H. GUGGENHEIMER, Minneapolis, USA

Solution: The given condition is equivalent to

$$(ABC)^2\,(BCA)^2 = (BCA)^2\,(ABC)^2\,,$$

that is, $(ABC)^2$ and $(BCA)^2$ commute. We use the fact that inversion preserves the relation of inverse points.

(1) If B and C intersect, we may invert with respect to one of the points of intersection, so that B and C become lines b and c. If the point of intersection of these lines is taken as the origin O, then the transformation bc (which is a rotation about O) may be written in the form $z' = kz$, where k has modulus 1. The circle A will become (in general) a circle a, so the inversion a may be written

$$z' - \alpha = \frac{r^2}{\bar{z} - \bar{\alpha}}\,.$$

We may then work out $(abc)^2$ and $(bca)^2$, and these commute in the following cases:
(i) $r^2 = \alpha\,\bar{\alpha}$, so a passes through O. Hence the circles A, B, C have a common point.
(ii) $k = 1$, so b and c coincide (a trivial case).
(iii) $\alpha = 0$, so a has centre O. Hence A cuts B and C orthogonally.
(iv) $k = -1$, so b and c cut orthogonally. Hence B and C cut orthogonally.
(If a is a line, it must pass through O, so A, B, C have a common point, as in (i).)