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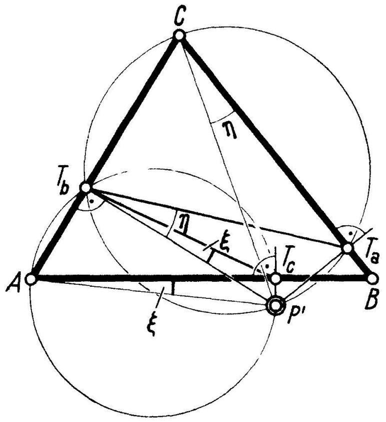
Die Punkte P', A, T_b, T_c und P', C, T_a, T_b liegen nach dem Satz von Thales je auf einem Kreis (siehe Figur 2). Bei allgemeiner Wahl von P' ergeben sich aus dem Peripheriewinkelsatz folgende gleiche Winkel: $\xi = \sphericalangle P'AB = \sphericalangle P'T_bT_c$, $\eta =$



Figur 2

$\sphericalangle P'CB = \sphericalangle P'T_bT_a$. Also unterscheiden sich ξ und η um $\sphericalangle T_aT_bT_c$. Genau dann, wenn T_a, T_b, T_c auf einer Geraden liegen, müssen ξ und η gleich sein, daher die vier Punkte A, B, C, P' auf einem Kreis liegen. Also: *Dann und nur dann liegen T_a, T_b, T_c auf einer Geraden, wenn P' auf den Umkreis des Dreiecks $\triangle$ fällt.*

In Verbindung mit der Deutung von (4) erhält man das Ergebnis: *Die gefährliche Fläche beim räumlichen Rückwärtsschnitt ist jener Drehzylinder, der die Ebene ε nach dem Umkreis des Dreiecks $\triangle$ schneidet.*

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On a Diophantine Equation

A. SCHINZEL and W. SIERPIŃSKI [2] [1] have recently showed that all solutions of the equation

$$(x^2-1)(y^2-1) = \left(\left(\frac{y-x}{2}\right)^2 - 1\right)^2 \tag{1}$$

in natural numbers x, y, $x < y$ are of the form $x = x_n$, $y = x_{n+1}$, $n = 0, 1, \ldots$, where $x_0 = 1$, $x_1 = 3$, $x_{n+2} = 6\,x_{n+1} - x_n$. Equation (1) is a special case of the equation

$$(x^2-1)(y^2-1) = (z^2-1)^2 \tag{2}$$

whose all solutions are still unknown (cf. [2]–[5]). Moreover, the only known solution of (2) which is not a solution of (1) is $x = 4$, $y = 31$, $z = 11$ ([5]; cf. [2], [3]).

The purpose of this note is to give all solutions in natural numbers t, x, y of the equation

$$(x^2-t^2)(y^2-t^2) = \left(\left(\frac{y-x}{2}\right)^2 - t^2\right)^2. \tag{3}$$

[1] Numbers in brackets refer to References, page 38.

We shall prove the following

Theorem. All solutions of the Equation (3) in distinct natural numbers $t, x, y, x < y$, are of the form

$$t = |m^2 - 2n^2| k, \quad x = (m^2 + 2n^2) k, \quad y = (3m^2 + 8mn + 6n^2) k, \tag{4}$$

where m, n, k are natural numbers.

Proof. As in [2] we observe that

$$(x^2 - t^2)(y^2 - t^2) - \left(\left(\frac{y-x}{2}\right)^2 - t^2\right)^2 = -\frac{1}{16}(x+y)^2(x^2 - 6xy + y^2 + 8t^2).$$

Thus (3) is equivalent to

$$x^2 - 6xy + y^2 + 8t^2 = 0,$$

which may be written in the form

$$(y - 3x)^2 = 8(x^2 - t^2). \tag{5}$$

First of all, $t < x$. Further, from (5) it follows that $4 \mid y - 3x$, so $y - 3x = 4z$, where $z > 0$, since otherwise $(y - x)/2 \leqslant x$ and from $t < x < y$ and (3) we get $(x^2 - t^2)(y^2 - t^2) \leqslant (x^2 - t^2)^2$, $y \leqslant x$, which is impossible.

Thus $z > 0$ and (5) can be written as

$$2z^2 + t^2 = x^2. \tag{6}$$

Consequently, every solution of (5) in natural numbers $t, x, y, x < y$, gives a solution of (6) in natural numbers t, x, z, where $4z = y - 3x$. On the other hand, if t, x, z is a solution of (6) in natural numbers, then the numbers $t, x, y = 3x + 4z$ are natural, $x < y$, and they form a solution of (5).

Thus, in order to find all solutions of (3) in natural numbers $t, x, y, x < y$ it suffices to know all solutions of (6) in natural numbers t, x, z and put $y = 3x + 4z$.

But all solutions of (6) are the following (cf. [1], p. 41):

$$t = |m^2 - 2n^2| k, \quad x = (m^2 + 2n^2) k, \quad z = 2mnk,$$

where m, n, k are natural numbers. If we put here $y = 3x + 4z$, we get the formulae (4), which completes the proof.

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