Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 24 (1969)
Heft: 4

Artikel: On Sc Functions

Autor: Tauber, S.

DOI: https://doi.org/10.5169/seals-26649

9. Schlussbetrachtung

Ziel der Arbeit ist es, an Beispielen eine Methode vorzuführen, die es gestattet, die Inhaltsmasszahlen gewisser hyperbolischer Rotationskörper zu ermitteln. Die Auswahl der Beispiele erfolgte nach zwei Gesichtspunkten. Einmal sollten alle in Teil 3 ausgewerteten Integrale bei der Bestimmung von Masszahlen tatsächlich vorkommen. Zum andern aber wollten wir vor allem solche Drehkörper behandeln, die in der euklidischen Geometrie ein Gegenstück haben. Es ist besonders interessant, in diesen Fällen den Übergang von der hyperbolischen zur euklidischen Formel durchzuführen. Wir zeigen das am Beispiel des Torus. In Formel (16) werden die vorkommenden hyperbolischen Funktionen in Reihen entwickelt:

$$V = \pi^2 k^3 \left(\frac{\bar{a}}{k} + \frac{1}{3!}\left(\frac{\bar{a}}{k}\right)^3 + \cdots\right)\left(1 + \frac{1}{2}\left(\frac{2\,\bar{R}}{k}\right)^2 + \cdots - 1\right)$$

Wächst jetzt k unbegrenzt, so ergibt sich das Torusvolumen der euklidischen Geometrie $V = \pi^2\, a \cdot 2\, R^2$.

H. ZEITLER, Weiden

On *Sc* Functions

Introduction

In this paper we prove that the Dirac Delta and all its derivatives can be represented by sequences of constructed discontinuous functions. Although this result is stated in [1] it is not formally proved.

We then prove that by using this definition of the n-th derivative of the Dirac Delta its Laplace Transform is s^n. This result again can be considered as "classical" (see for example [3]) but is not proved either.

We feel that although the results are known the approach is new and our proof is rigorous which justifies the contents of this paper.

Definition of the *n*-th Derivative of a Function

Let $V = [v_1, v_2, \ldots, v_n]$ be an n-dimensional vector. We say that the vector tends *basewise* to zero if the components v_k tend to zero *successively*. We write symbolically

$$V \underset{B}{\longrightarrow} 0\,. \tag{1}$$

Geometrically speaking this means that the end point of V describes a polygonal line whose sides are parallel to the axes of the basis.

We shall use the notation $\Pi V = \prod_{m=1}^{n} v_m$ for the product of the components of the vector.

Let $f(t) \in C^n[b, c]$ be the class of functions that are defined and continuous as well as their derivatives up to and including the order n for $b \leqq t \leqq c$. Let a_k, $k = 1, 2, \ldots, n$, be such that $(t + \alpha_{h,n}) \in [b, c]$, where $\alpha_{h,n}$ represents the sum of any h of the n

numbers a_k, and $b + \varepsilon \leqq t \leqq c - \varepsilon$, ε being a given positive number. Clearly

$$\left.\begin{aligned} Df(t) &= \lim_{a_1 \to 0} (1/a_1) [f(t + a_1) - f(t)] , \\ D^2f(t) &= \lim_{\substack{a_1 \to 0 \\ a_2 \to 0}} (1/a_1 a_2) [f(t + a_1 + a_2) - f(t + a_1) - f(t + a_2) + f(t)] , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ D^nf(t) &= \lim_{\substack{a \to 0 \\ B}} (1/\Pi\, \boldsymbol{a}) \sum_{k=0}^{n} (-1)^{n+k} \varphi_k , \end{aligned}\right\} \quad (2)$$

where,

$$\boldsymbol{a} = [a_1, a_2, \ldots, a_n] , \qquad \varphi_k = \sum_{m=1}^{\binom{n}{k}} f(t + \alpha_{k,m}) , \qquad \alpha_{k,m} = a_{s_1} + a_{s_2} + \ldots + a_{s_k} ,$$

$$s_i, s_j = 1, 2, \ldots, n , \qquad s_i \neq s_j , \qquad i, j = 1, 2, \ldots, k .$$

Since φ_k is symmetric with respect to the a_k's, $D^nf(t)$ is independent of the order in which the different a_k's tend to zero, this is why no specific order is necessary when a vector tends basewise to zero.

If in $\boldsymbol{a} \underset{B}{\to} 0$ we make a change of variables in the a_k's, change defined by

$$\boldsymbol{b} = H\, \boldsymbol{a} , \tag{3}$$

where H is a $n \times n$ matrix, this change of variables corresponds to a rotation of the reference system. When $\boldsymbol{a} \underset{B}{\to} 0$, the last leg of the polygonal line described by the end-point of $\boldsymbol{a}$ is a straight line. With respect to the new reference system, when $\boldsymbol{a}$ tends to zero basewise then $\boldsymbol{b}$ tends to zero, although *not* basewise. Under these conditions, considering (3) all the b_k's, components of $\boldsymbol{b}$ will tend to zero *simultaneously.*

We may thus assume without loss of generality that in (2) all the a_k's tend to zero simultaneously. In addition we may assume that all the a_k's are equal to a. This corresponds to a special choice of the matrix H in (3) that would make all the b_k's equal.

It follows that we can define the n-th derivative of the function $f(t)$ by the expression

$$D^nf(t) = \lim_{a \to 0} a^{-n} \sum_{k=0}^{n} (-1)^{k+n} \binom{n}{k} f(t + k\, a) . \tag{4}$$

If $0 \leqq \theta \leqq 1$, and $0 \leqq \beta_k \leqq 1$, $k = 1, 2, \ldots, n$, we can write according to the mean-value theorem

$$f[f + (k + \theta)\, a] = f(t + k\, a) + a\, \theta\, Df[t + (k + \beta_k\, \theta)\, a] ,$$

so that by substitution into (4) we obtain

$$D^nf(t) = \lim_{a \to 0} a^{-n} \sum_{k=0}^{n} (-1)^{n+k} [f[t + \theta)\, a] - a\, \theta\, Df[t + (k + \beta_k\, \theta)\, a]] ,$$

where the second term in the sum tends to zero with a.

We can thus write for the n-th derivative of $f(t)$

$$D^n f(t) = \lim_{a \to 0} (-1)^n a^{-n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f[t + (k + \theta) a]. \tag{5}$$

This is the form we are going to use in the next section.

Remark. (5) can easily be checked by writing

$$f[t + (k + \theta) a] = f(t) + \sum_{m=1}^{n-1} (m!)^{-1} a^m (k + \theta)^m D^m f(t)$$
$$+ (k + \theta)^n a^n D^n f[t + \beta_k (k + \theta) a]/n!, \quad 0 \leqq \beta_k \leqq 1,$$

thus substituting into (5)

$$D^n f(t) = \lim_{a \to 0} (-1)^n a^{-n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} [f(t) + \sum_{m=0}^{n-1} (m!)^{-1} a^m (k + \theta)^m D^m f(t)$$
$$+ (n!)^{-1} a^n (k + \theta)^n D^n f[t + \beta_k (k + \theta) a]]$$
$$= \lim_{a \to 0} (-1)^n a^{-n} \Big\{ f(t) \sum_{k=0}^{n} (-1)^k \binom{n}{k} + \sum_{m=1}^{n-1} (m!)^{-1} a^m D^m f(t) \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k + \theta)^m$$
$$+ (n!)^{-1} a^n D^n f[t + \beta_k (k + \theta) a] \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k + \theta)^n \Big\}.$$

Since
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (k + \theta)^m = (-1)^n n! \, \delta_n^m,$$

where δ_n^m is the Kronecker Delta, it follows that all the terms on the right hand side, except the last one, cancel out, so that

$$D^n f(t) = \lim_{a \to 0} (-1)^n a^{-n} (n!)^{-1} a^n D^n f[t + \beta_k (k + \theta) a] \, n! \, (-1)^n,$$

which clearly is an identity.

3. The accordeon function

We shall use the following notation for the Heaviside-step function:

$$u(t - T) = \begin{cases} 0 & \text{for} \quad t < T, \\ 1 & \text{for} \quad t > T, \end{cases} \qquad f(T^-) = 0, \quad f(T^+) = 1,$$

so that for $T < \theta$, and $f(t) \in C^\circ[T, \theta]$,

$$\varphi(t) = f(t) [u(t - T) - u(t - \theta)] = \begin{cases} 0 & \text{for} \quad t < T, \\ f(t) & \text{for} \quad T < t < \theta, \\ 0 & \text{for} \quad \theta < t, \end{cases}$$

$$\varphi(T^-) = 0, \quad \varphi(T^+) = f(T^+), \quad \varphi(\theta^-) = f(\theta^-), \quad \varphi(\theta^+) = 0.$$

Under these conditions, with $a > 0$, we define the following *accordeon function:*

$$\left.\begin{aligned} \mathrm{Ac}(t, n, a) &= a^{-n-1} \sum_{m=0}^{n} (-1)^m \binom{n}{m} [u(t - m a) - u(t - (m+1) a)], \\ &= a^{-n-1} \sum_{m=0}^{n+1} (-1)^m u(t - m a) \left[\binom{n}{m} + \binom{n}{m-1}\right] \\ &= a^{-n-1} \sum_{m=0}^{n+1} (-1)^m \binom{n+1}{m} u(t - m a), \end{aligned}\right\} \tag{6}$$

where use has been made of the classical properties of the binomial coefficients and $\binom{n}{-1} = 0$.

If we let $a \to 0$ in (6) we obtain a generalized function in the sense of MIKUSINSKI (cf. [2] and [3]). It is easier in this case not to use the notion of distribution in the sense of SCHWARTZ. We shall call this generalized function a *squeezed accordeon* and shall write

$$\lim_{a \to 0} \mathrm{Ac}(t, n, a) = \mathrm{Sc}(t, n). \tag{7}$$

For any function $f(t)$ defined and continuous over a sufficiently large neighbourhood of $t = 0$ we have, using the classical notation for the inner product

$$\left.\begin{aligned} \langle \mathrm{Sc}(t, n), f(t) \rangle &= \int_{-\infty}^{+\infty} f(t)\, \mathrm{Sc}(t, n)\, dt \\ &= \int_{-\infty}^{+\infty} f(t) \left[\lim_{a \to 0} a^{-n-1} \sum_{m=0}^{n+1} (-1)^m u(t - m a) \binom{n+1}{m}\right] dt. \end{aligned}\right\} \tag{8}$$

Since the integration is independent of a and of m we can change the order of the operations and write, using (6)

$$\langle \mathrm{Sc}(t, n), f(t) \rangle = \lim_{a \to 0} a^{-n-1} \sum_{m=0}^{n} (-1)^m \binom{n}{m} \int_{-\infty}^{+\infty} f(t) [u(t - m a) - u(t - (m+1) a)]\, dt.$$

The integral can be written

$$\int_{-\infty}^{+\infty} f(t) [u(t - m a) - u(t - (m+1) a)]\, dt = \int_{ma}^{(m+1)a} f(t)\, dt = a f[(m + \theta_m) a]$$

$$= a f[(m + \theta + \beta_m) a] = a [f[(m + \theta) a] + a \beta_m Df[(m + \theta + \eta_m \theta_m) a]],$$

where $0 \leqq \theta_m \leqq 1$, θ is a fixed number such that

$$0 \leqq \theta < \min(\theta_1, \theta_2, \ldots, \theta_n), \quad \theta_m = \theta + \beta_m, \quad \text{thus } 0 \leqq \beta_m \leqq 1, \quad 0 \leqq \eta_m \leqq 1.$$

It follows that

$$\langle \mathrm{Sc}(t, n), f(t) \rangle = \lim_{a \to 0} a^{-n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} \{f[(m+\theta) a] + a \beta_m Df[(m + \theta + \eta_m \beta_m) a]\},$$

where the second term in the sum tends to zero with a so that

$$\langle \mathrm{Sc}(t, n), f(t) \rangle = \lim_{a \to 0} a^{-n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f[(m+\theta) a],$$

which according to (5) gives

$$\langle \mathrm{Sc}(t, n), f(t) \rangle = (-1)^n D^n f(0). \tag{9}$$

(9) shows that $\mathrm{Sc}(t, n)$ is identical to the n-th derivative of the Dirac Delta, i.e. $\mathrm{Sc}(t, n) = \delta^{(n)}(t)$, as it is usually considered (cf. [4]). In particular for $n = 0$,

$$\langle \mathrm{Sc}(t, 0), f(t) \rangle = f(0) = \langle \delta(t), f(t) \rangle, \tag{10}$$

i.e. $\mathrm{Sc}(t, 0) = \delta(t)$, the Dirac Delta.

4. Laplace Transforms

We clearly have

$$\mathcal{L}[\mathrm{Sc}(t, n)] = \int_{0^+}^{+\infty} e^{-st} \mathrm{Sc}(t, n)\, dt, \tag{11}$$

where the integration starts on the positive side of zero. Thus

$$\mathcal{L}[\mathrm{Sc}(t, n)] = \int_{0^+}^{+\infty} e^{-st} \left[\lim_{a \to 0} a^{-n-1} \sum_{m=0}^{n+1} (-1)^m \binom{n+1}{m} u(t - m a)\right] dt.$$

Since the integration is independent of a and m we may change the order of operations, i.e.

$$\begin{aligned}
\mathcal{L}[\mathrm{Sc}(t, n)] &= \lim_{a \to 0} a^{-n-1} \sum_{m=0}^{n+1} \binom{n+1}{m} (-1)^m \int_{0^+}^{+\infty} e^{-st} u(t - m a)\, dt \\
&= \lim_{a \to 0} a^{-n-1} \sum_{m=0}^{n+1} \binom{n+1}{m} (-1)^m [u(t - m a)] \\
&= \lim_{a \to 0} a^{-n-1} \sum_{m=0}^{n+1} \binom{n+1}{m} (-1)^m e^{-mas}/s \\
&= \lim_{a \to 0} (1 - e^{-as})^{n+1}/s\, a^{n+1} = s^n.
\end{aligned}$$

It follows that

$$\mathcal{L}[\mathrm{Sc}(t, n)] = s^n,$$

or,

$$\mathrm{Sc}(t, n) = \delta^{(n)}(t) = \mathcal{L}^{-1} s^n.$$

Thus the Sc (t, n) function, i.e. the n-th derivative of the Dirac Delta is the inverse Laplace Transform of s^n. This result is considered classical and can be found for example in [4]. S. TAUBER, Portland State University USA

REFERENCES

[1] VAN DER POL and BREMMER, *Operational Calculus* (Cambridge Univ. Press, Cambridge 1964).
[2] J. MIKUSINSKI, *Operational Calculus* (Pergamon Press, N.Y. 1959).
[3] A. H. ZEMANIAN, *Distribution Theory and Transform Analysis* (McGraw-Hill, N.Y. 1965).
[4] W. KAPLAN, *Operational Methods for Linear Systems* (Addison Wesley, Reading Mass. 1962).

Aufgaben

Aufgabe 577. K. RADZISZEWSKI (Ann. Univ. Marie Curie-Sklodowska *A* 10, 57–59, 1956) hat bewiesen: Es sei P der Flächeninhalt des Rechtecks, das einem gegebenen Oval umschrieben ist und das eine Seite in der Richtung θ hat. Der Flächeninhalt des Ovals sei S. Dann ist

$$\frac{4}{\pi} S \leqslant \bar{P} = \frac{1}{2\pi} \int_0^{2\pi} P \, d\theta$$

mit Gleichheit nur für den Kreis. Man beweise: Es sei S^* der Flächeninhalt der Fusspunktkurve des Ovals für einen beliebigen inneren Punkt. Dann ist

$$\bar{P} \leqslant \frac{4}{\pi} S^*$$

mit Gleichheit nur, wenn das Oval durch eine Rotation von 90° in sich übergeführt werden kann. S^* hat ein einziges Minimum, wenn der Aufpunkt im Inneren variiert. Für glatte Ovale wird das Minimum im Krümmungsschwerpunkt angenommen.

H. GUGGENHEIMER, Polytechnic Institute of Brooklyn, USA

Lösung des Aufgabenstellers: Das Oval habe die Stützfunktion $h(\theta)$, gegeben als Funktion des Tangentenwinkels. Dann ist

$$\bar{P} = \frac{2}{\pi} \int_0^{2\pi} h(\theta)\, h\left(\theta + \frac{\pi}{2}\right) d\theta .$$

Wenn der Nullpunkt des Koordinatensystems ein innerer Punkt des Ovals ist, so ist die Fusspunktkurve die Kurve deren Polargleichung $r(\phi) = h(\theta)$ ist, $\theta = \phi + \pi/2$. Daher ist

$$S^* = \frac{1}{2} \int_0^{2\pi} h^2(\theta) \, d\theta .$$

Die gefragte Ungleichung folgt sofort aus der Schwarzschen Ungleichung für das Integral P. Gleichheit besteht, wenn $h(\theta) = h\,(\theta + \pi/2)$ für alle θ.

Eine Translation des Aufpunktes resultiert in einer Änderung der Stützfunktion

$$h(\theta) \to h(\theta) + a \cos\theta + b \sin\theta .$$