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Kleine Mitteilungen

The Missing Seventh Circle

(To the memory of Dr. A. Aeppli, of Zürich)

The problem of Apollonius, to construct circles tangent to three given circles C_1, C_2, C_3, has given rise to many papers and much discussion. In 1896 MUIRHEAD [2] stated that nobody before STOLL [6], in a paper published in 1873, had ever discussed the possibility of the circles C_i being so specialized that the number of contact circles varies. Stoll remarks that the inspiration for his paper was Prof. Sturm, who pointed out that in the general Apollonius configuration the number of contact circles is either 8, 4 or 0, and that Prof. Sturm encouraged him to investigate all possible cases. Stoll does observe that if the circles C_i are allowed to touch, the number of tangent circles is never 1 or 7. He allows the possibility of a tangent circle being one of the circles C_i, assuming, correctly, that since the problem is an algebraic one, a circle may be considered to touch itself. For example, if the three circles C_i touch each other, the maximum number of contact circles distinct from the C_i is 2, and since each one of the C_i satisfies the conditions of the problem, we have a total of 5 tangent circles. For the Descartes formula connected with this case see AEPPLI [1] and PEDOE [4]. Again, Stoll recognizes that a circle of zero radius (a point-circle) lying on a circle can be considered to touch this circle, so that if the three circles C_i pass through a point O, there are 4 circles of non-zero radius which touch the C_i, and to these must be added the point-circle O, so that there are 5 tangent circles.

But neither Stoll nor Muirhead noticed that it is impossible to specialize the circles C_i so that the number of tangent circles is 7, and the proof of this curious result is the purpose of this paper. It is possible to specialize the C_i so that the number of tangent circles is 0, 1, 2, 3, 4, 5 and 6, and 8 is the maximum finite number of tangent circles, but the mystic number 7 is absent. It was when I was engaged in making up exercises for a forthcoming book (PEDOE [5]) that I noticed that I could not specialize the C_i to produce 7 contact circles, and investigation revealed why this is so. Multiplicity problems are of not too frequent occurrence in geometry, and the Apollonius problem is even more interesting than was at first suspected.

The circle C_0 which is orthogonal to each of the C_i is uniquely defined, unless the C_i belong to a coaxal system (pencil) of circles. In the latter case the only tangent circles are two point-circles, in the case when the coaxal system is of the intersecting type. The circle C_0 plays a very special rôle with regard to the C_i since inversion (transformation by reciprocal radii) in C_0 maps each C_i onto itself, and maps a tangent circle C onto a tangent circle C'. When there are 8 tangent circles (which may be called the general case) these can be split into 4 pairs. We shall call the circles in a pair conjugate circles. (For all this, proved algebraically, see PEDOE [3]). If we wish to specialize the C_i so that there are only 7 tangent circles, the specialization must aim at making a pair of conjugate circles identical, since if two tangent circles which are not conjugate become identical the conjugates also become identical, and the number of tangent circles would reduce to 6, at most.

We therefore specialize the C_i so that a conjugate pair C and C' become the same circle D, say. This means that inversion in C_0 maps the tangent circle D onto itself. If this is the case, D must be orthogonal to C_0. We therefore find ourselves with three circles C_i, a circle C_0 orthogonal to the C_i, and a circle D which touches the C_i and is also orthogonal to C_0. We show that this means that two of the C_i touch each other.

Invert with respect to a centre of inversion on C_0. We obtain three circles C'_i, with diameters which lie along the line C'_0. These three circles are touched by a circle D' whose diameter also lies along C'_0. If two circles with diameters along the same line touch at a point not on this line, they have the same centre, and must therefore coincide. If the circles are distinct contact can only take place at an endpoint of a diameter. Since D' has only 2 points of intersection with the line C'_0, and has to touch each of C'_1, C'_2 and C'_3 at a point on C_0', the three points of contact cannot be distinct. Hence at least two of the circles C_i' intersect C_0' at the same point. That is, at least two of the circles C_i' touch each other. But if at least two of the circles C_i touch each other, the number of circles tangent to the three C_i is readily seen to be 6, at most.

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Die Jakobische Achsenkonstruktion einer Ellipse

Anregung zu dieser kurzen Mitteilung ist die Arbeit [1].

Eine Ellipse ε (Mittelpunkt $\boldsymbol{M}$) sei gegeben durch zwei konjugierte Halbmesser $\overline{\boldsymbol{M}\boldsymbol{A}} = \boldsymbol{a}_1$ und $\overline{\boldsymbol{M}\boldsymbol{B}} = \boldsymbol{b}_1$. Die Richtungen der Hauptachsen lassen sich folgenderweise finden. Es sei $\boldsymbol{g}$ die Gerade durch $\boldsymbol{B}$ parallel zu $\boldsymbol{M}\boldsymbol{A}$ und es seien γ_1 (Mittelpunkt $\boldsymbol{M}_1$) und γ_2 (Mittelpunkt $\boldsymbol{M}_2$) die beiden Kreise (Halbmesser $\boldsymbol{a}_1$) welche $\boldsymbol{g}$ in $\boldsymbol{B}$ berühren. Dann gibt es eine Affinität $\mathfrak{a}_k$ die γ_k in ε überführt ($k = 1, 2$). Die übliche Konstruktion für das invariante Rechtwinkelpaar von $\mathfrak{a}_k$ führt nun zu den Richtungen der Hauptachsen von ε. Der Zusammenhang mit der Jakobischen Achsenkonstruktion tritt aus der Figur in aller Klarheit hervor. Setzt man $\sphericalangle\, \boldsymbol{A}\boldsymbol{M}\boldsymbol{B} = \varphi$ und sind $2\,\boldsymbol{a}$ und $2\,\boldsymbol{b}$ die Längen der Hauptachsen ($\boldsymbol{a} > \boldsymbol{b}$), dann gilt in $\Delta\, \boldsymbol{M}\boldsymbol{M}_1\boldsymbol{B}$:

$$\overline{M M_1}^2 = \boldsymbol{a}_1^2 + \boldsymbol{b}_1^2 - 2\,\boldsymbol{a}_1\,\boldsymbol{b}_1 \sin\varphi = (\boldsymbol{a} - \boldsymbol{b})^2\,;$$

daher: $\overline{\boldsymbol{M}\boldsymbol{M}_1} = \boldsymbol{a} - \boldsymbol{b}$. Analog findet man $\overline{\boldsymbol{M}\boldsymbol{M}_2} = \boldsymbol{a} + \boldsymbol{b}$.

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LITERATURVERZEICHNIS

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