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Zweierdrehung und fand sie auch sofort: die Drehungsachse ist die Verbindungslinie MN der Mitten von AD und BC, die sich in Figur 3 als Punkt projiziert. Dass MN eine Symmetrieachse der Figur ist, war mir anschaulich evident.

Zwei kleine Aufgaben blieben übrig. Zunächst musste ich beweisen, dass $MN \perp AD$ und $MN \perp BC$ ist. Wenn man in der Raumgeometrie beweisen will, dass zwei Geraden senkrecht sind, so macht man das häufig so, dass man zeigt, dass die eine in einer Ebene senkrecht zur anderen liegt. Ich dachte mir also eine Ebene durch M senkrecht zu AD. Ich hatte zu zeigen, dass nicht nur M, sondern auch N in dieser Ebene liegt. Die Ebene durch die Mitte M senkrecht zu AD ist der Ort aller Punkte, die von A und D gleich weit entfernt sind; also hatte ich nur noch $NA = ND$ zu beweisen und ebenso $MB = MC$. Der Beweis war leicht.

So erhielt ich ohne neue Einfälle eine Zweierdrehung, die A in D und B in C überführt. Nun war noch zu zeigen, dass E auf der Drehungsachse liegt, also bei der Drehung fest bleibt. Wie schon erwähnt, folgt diese Behauptung sofort aus der Tatsache, dass E gleich weit entfernt ist von A und D und ebenso von B und D.

Der Rest des Beweises ergab sich ganz von selbst ohne neue Einfälle, wie unter II gezeigt wurde.

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Triangular Covers for Closed Curves of Constant Length

1. We call a compact, convex (plane) set S a *translation* [*displacement*] *cover* for the family $\mathfrak{C}$ of all closed (plane) curves of fixed length L if for each curve $\Gamma \in \mathfrak{C}$ there is a translation τ [displacement[1]) δ] such that $\tau(\Gamma) \subseteq S$ [$\delta(\Gamma) \subseteq S$]. In this note we describe the triangular translation and displacement covers for $\mathfrak{C}$ of prescribed shape that have least area.

It is a consequence of our results that the smallest triangular translation cover and the smallest triangular displacement cover are both equilateral, the first with side $2L/3$ and the second with side $L\sqrt{3}/\pi$.

Along the way we obtain a sharpened version of a theorem of H. G. EGGLESTON's on the thickness of a triangle that is circumscribed about a curve in $\mathfrak{C}$ and similar inequalities for the diameter and area of such triangles.

The translation theory, which depends on a well-known property of the orthic triangle, is developed in section 2. The displacement theory depends on an inequality of Eggleston's and is summarized in section 3.

2. We begin by recalling some formulas from the trigonometry of triangles. Let p, h_a, t, d, T, r and q be the perimeter, altitude to side a, thickness (minimal altitude), diameter (maximal side), area, inradius, and the perimeter of the orthic (pedal) triangle, respectively, of a triangle (ABC). Then

$$a = p\,\frac{\sin\alpha}{\sin\alpha + \sin\beta + \sin\gamma}, \qquad h_a = p\,\frac{\sin\beta\sin\gamma}{\sin\alpha + \sin\beta + \sin\gamma},$$

1) A *displacement*, also called a rigid motion, is an orthogonal map of the plane that preserves orientation. For details see [5; 32].

$$T = \frac{p^2}{2}\,\frac{\sin\alpha\sin\beta\sin\gamma}{(\sin\alpha+\sin\beta+\sin\gamma)^2}\,, \qquad r = p\,\frac{\sin\alpha\sin\beta\sin\gamma}{(\sin\alpha+\sin\beta+\sin\gamma)^2}\,,$$

and, when (ABC) is acute,

$$q = 2p\,\frac{\sin\alpha\sin\beta\sin\gamma}{\sin\alpha+\sin\beta+\sin\gamma}\,.$$

For convenience, we define

$$\operatorname{Sin} x = \begin{cases} \sin x & \text{for } 0 \leqslant x \leqslant \pi/2 \\ 1 & \text{for } \pi/2 \leqslant x \leqslant \pi\,. \end{cases}$$

Then it follows that

$$2p\,\frac{\operatorname{Sin}\alpha\operatorname{Sin}\beta\operatorname{Sin}\gamma}{\sin\alpha+\sin\beta+\sin\gamma} = \begin{cases} q & \text{if } (ABC) \text{ is acute} \\ 2t & \text{if } (ABC) \text{ is not acute.} \end{cases} \tag{1}$$

A key role in the argument is played by the minimum property of the orthic triangle called Fagnano's problem (see [1; 20–21]): There is a unique triangle with minimum perimeter inscribed in any given triangle (ABC); when (ABC) is acute, it is the orthic triangle, and when (ABC) is not acute, it is the shortest altitude segment covered twice (which we call the 'thickness needle' of (ABC) and regard as a triangle for linguistic convenience). A beautiful geometric proof found by L. FEJÉR is given in [4; 30–35] (see also [3; 179–180]).

Theorem 1. Let (ABC) be a triangle circumscribed about a closed curve Γ of length L, and suppose the notation is arranged so that α is a maximal angle. Then

(a) $$p \leqslant \frac{L}{2}\,\frac{\sin\alpha+\sin\beta+\sin\gamma}{\operatorname{Sin}\alpha\operatorname{Sin}\beta\operatorname{Sin}\gamma}\,,$$

(b) $$t \leqslant \frac{L}{2}\,\frac{1}{\operatorname{Sin}\alpha}\,,$$

(c) $$d \leqslant \frac{L}{2}\,\frac{1}{\operatorname{Sin}\alpha}\,\frac{\sin\alpha}{\sin\beta\sin\gamma}\,,$$

(d) $$T \leqslant \frac{L^2}{8}\,\frac{1}{\operatorname{Sin}^2\alpha}\,\frac{\sin\alpha}{\sin\beta\sin\gamma}\,.$$

In each case, the equality holds if and only if the curve Γ is the orthic triangle of (ABC), when (ABC) is acute, and the thickness needle of (ABC), when (ABC) is not acute.

Proof. Since the inequalities (b), (c) and (d) result from substituting (a) into the formulas given at the beginning of the section, we must prove only (a). Let (ABC) be circumscribed about Γ, let X, Y and Z be points of Γ that lie on the (closed) sides BC, CA and AB respectively of (ABC), and let P be the perimeter of (XYZ). Evidently $L \geqslant P$, with equality if and only if the curve Γ and the triangle (XYZ) coincide. (There is an obvious linguistic gloss when (XYZ) degenerates to a needle.) By Fagnano's problem and (1),

$$L \geqslant P \geqslant 2p\,\frac{\operatorname{Sin}\alpha\operatorname{Sin}\beta\operatorname{Sin}\gamma}{\sin\alpha+\sin\beta+\sin\gamma}\,,$$

which is the desired inequality (a). The cases of equality are immediate.

EGGLESTON [2; 150–153] proved that the thickness t of a triangle (ABC) circumscribed about a closed curve Γ of length L is at most $L/\sqrt{3}$, with equality if and only if Γ and (ABC) are both equilateral triangles and the vertices of Γ are at the midpoints of the sides of (ABC). This is a consequence of inequality (b), because $\alpha \geqslant \pi/3$.

Now we can characterize triangular translation covers.

Theorem 2. A triangular region (ABC) with perimeter p and maximal angle α is a translation cover for $\mathfrak{C}$ if and only if

$$p \geqslant \frac{L}{2} \frac{\sin\alpha + \sin\beta + \sin\gamma}{\operatorname{Sin}\alpha \operatorname{Sin}\beta \operatorname{Sin}\gamma}, \tag{2}$$

and the least area $T_\tau(ABC)$ of all triangular translation covers for $\mathfrak{C}$ similar to (ABC) is

$$T_\tau(ABC) = \frac{L^2}{8} \frac{1}{\operatorname{Sin}^2\alpha} \frac{\sin\alpha}{\sin\beta \sin\gamma}.$$

Proof. Let $\Gamma \in \mathfrak{C}$, and by a translation assume that Γ lies in $\angle BAC$ and has points of contact on the (closed) rays $\boldsymbol{AB}$ and $\boldsymbol{AC}$. Let B' and C' be points on $\boldsymbol{AB}$ and $\boldsymbol{AC}$ respectively so that the line $B'C'$ is a support line of Γ parallel to BC and $(AB'C')$ surrounds Γ. Let p' be the perimeter of $(AB'C')$. Then by (2) and inequality (a) of theorem 1,

$$p \geqslant \frac{L}{2} \frac{\sin\alpha + \sin\beta + \sin\gamma}{\operatorname{Sin}\alpha \operatorname{Sin}\beta \operatorname{Sin}\gamma} \geqslant p',$$

so that (ABC) surrounds $(AB'C')$ and hence the curve Γ. The converse is clear, and the formula for the minimum area is immediate because (2) gives the minimum perimeter.

Corollary 3. The triangular translation cover for $\mathfrak{C}$ that has least area is the equilateral triangle with side $2L/3$. This equilateral triangle also has the least perimeter among all triangular translation covers for $\mathfrak{C}$.

Proof. It is immediate from the formula for $T_\tau(ABC)$ that when the largest angle α is held fixed, the minimum area occurs when (ABC) is isosceles: $\beta = \gamma = (\pi - \alpha)/2$. But for such triangles, $T_\tau(ABC)$ is an increasing function of α, so its minimum occurs when α is as small as possible, namely, when $\alpha = \pi/3$.

3. H. G. EGGLESTON proved inequality (a) of the following theorem in [2; 157–158].

Theorem 4. Let Γ be a closed curve of length L and $(A'B'C')$ a given triangle with angles α, β and γ, the notation being arranged so that α is a maximal angle. Then there exists a triangle (ABC) similar to $(A'B'C')$ circumscribed about Γ so that all the following inequalities hold:

(a) $$p \leqslant \frac{L}{2\pi} \frac{(\sin\alpha + \sin\beta + \sin\gamma)^2}{\sin\alpha \sin\beta \sin\gamma},$$

(b) $$t \leqslant \frac{L}{2\pi} \frac{\sin\alpha + \sin\beta + \sin\gamma}{\sin\alpha},$$

(c) $$d \leqslant \frac{L}{2\pi} \frac{\sin\alpha + \sin\beta + \sin\gamma}{\sin\beta \sin\gamma},$$

(d) $$T \leqslant \frac{L^2}{8\pi^2} \frac{(\sin\alpha + \sin\beta + \sin\gamma)^2}{\sin\alpha \sin\beta \sin\gamma}.$$

These inequalities are sharp in the sense that for given $(A'B'C')$ there are curves Γ and circumscribed triangles (ABC) similar to $(A'B'C')$ for which all the equalities hold.

Proof. Part (a) is Eggleston's inequality, and the remaining inequalities result from substituting (a) into the formulas given at the beginning of section 2. Since (a) says precisely that $2\pi r \leqslant L$, the theorem asserts the existence of a circumscribing triangle similar to the given triangle $(A'B'C')$ whose incircle has circumference at most L. In particular, the equality in (a) (and consequently also in the other inequalities) holds when Γ is a circle with circumference L (and possibly for other curves as well).

It is of some interest to compare the bounds for p given by the first parts of theorems 1 and 4. Eggleston's bound is clearly smaller; the following sharp inequality holds in the other direction:

$$\frac{L}{2}\,\frac{\sin\alpha+\sin\beta+\sin\gamma}{\operatorname{Sin}\alpha\,\operatorname{Sin}\beta\,\operatorname{Sin}\gamma} < \frac{\pi}{2}\,\frac{L}{2\pi}\,\frac{(\sin\alpha+\sin\beta+\sin\gamma)^2}{\sin\alpha\sin\beta\sin\gamma}\,.$$

Indeed, when (ABC) is acute,

$$\frac{\text{R.H.S.}}{\text{L.H.S.}} = \frac{1}{2}\,(\sin\alpha+\sin\beta+\sin\gamma) > 1\,,$$

and when (ABC) is not acute and has largest angle α,

$$\frac{\text{R.H.S.}}{\text{L.H.S.}} = \frac{1}{2}\left(1+\frac{\cos\frac{\beta-\gamma}{2}}{\cos\frac{\beta+\gamma}{2}}\right) > 1\,.$$

The bound is sharp, because the ratio has limit 1 in degenerate cases.

The analogous result to theorem 2 for triangular displacement covers is immediate from Eggleston's inequality:

Theorem 5. A triangular region (ABC) with perimeter p is a displacement cover for $\mathfrak{C}$ if and only if

$$p \geqslant \frac{L}{2\pi}\,\frac{(\sin\alpha+\sin\beta+\sin\gamma)^2}{\sin\alpha\,\sin\beta\,\sin\gamma}\,, \tag{3}$$

and the least area $T_\delta(ABC)$ of all triangular displacement covers similar to (ABC) is

$$T_\delta(ABC) = \frac{L^2}{8\pi^2}\,\frac{(\sin\alpha+\sin\beta+\sin\gamma)^2}{\sin\alpha\,\sin\beta\,\sin\gamma}\,.$$

The proof is entirely analogous to the proof of theorem 2, using a displacement and inequality (a) of theorem 3 in place of the translation and inequality (a) of theorem 1. Geometrically, the theorem asserts that a triangle is a displacement cover for $\mathfrak{C}$ if and only if it contains a circle with circumference L.

Corollary 6. The triangular displacement cover for $\mathfrak{C}$ that has least area is the equilateral triangle with side $L\sqrt{3}/\pi$. This equilateral triangle also has the least perimeter among all triangular displacement covers for $\mathfrak{C}$.

Proof. A minimal triangular displacement cover must be circumscribed about a circle with circumference L. The smallest such triangle is equilateral, and since this equilateral triangle is a displacement cover, the assertion follows.

Added in proof: Shortly after this paper was submitted, the author found the problem of characterizing the best triangular covers for closed curves of unit length (solved in theorems 2 and 5) posed in H. T. CROFT's mimeographed 1969 'Addenda' to his well-known 1967 'Research Problems'.

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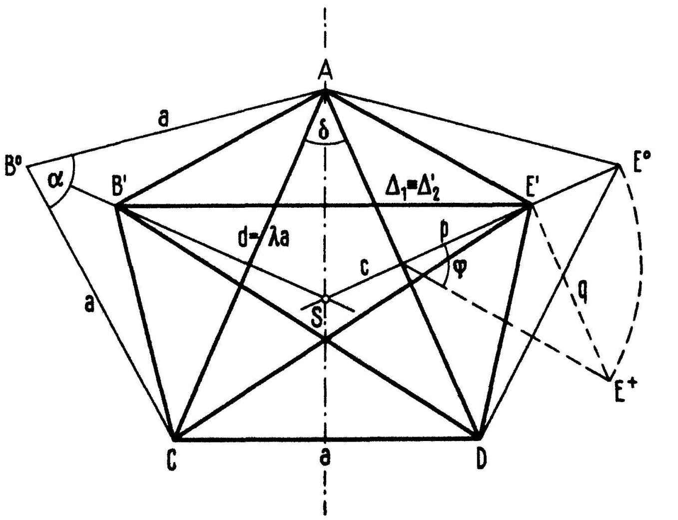
Kleine Mitteilungen

Zu einem Satz über räumliche Fünfecke

Der Satz lautet: *Ein räumliches Fünfeck ABCDE, in dem alle Seiten gleich a und alle Winkel gleich α sind, ist eben.* Die Rolle, die «Einfall und Überlegung» beim Beweis dieses Satzes spielten, hat B. L. VAN DER WAERDEN aufgezeigt[1]). Der gruppentheoretische Aspekt gibt dem Beweis von van der Waerden seine besondere Eleganz. Der Satz lässt sich aber auch mit den Methoden der elementaren Schulgeometrie gewinnen, wobei allerdings etwas gerechnet werden muss.



Es ist klar, dass alle Diagonalen die gleiche Länge $d = \lambda a$ haben. Wir betrachten eine Normalprojektion des Fünfecks, von dem das Dreieck ACD in der Projektionsebene liegt. Die Umklappungen B^0 und E^0 sind eindeutig bestimmt. δ sei der Winkel zweier aneinanderstossender Diagonalen, φ der Neigungswinkel der Ebenen ABC und AED gegen die Projektionsebene; wegen $CE \doteq DB = \lambda a$ ist die Mittelsenkrechte zu CD Symmetrieachse der (ebenen) Figur. Man bestätigt sofort:

$$\sin\frac{\alpha}{2} = \frac{\lambda}{2}\,;\ \cos\frac{\alpha}{2} = \frac{\sqrt{4-\lambda^2}}{2}\,;\ \sin\frac{\delta}{2} = \frac{1}{2\lambda}\,;\ \cos\frac{\delta}{2} = \frac{\sqrt{4\lambda^2-1}}{2\lambda}\,.$$

1) Dieses Heft S. 73-78.