

Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 28 (1973)
Heft: 5

Artikel: Minimum area of circumscribed polygons
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DOI: <https://doi.org/10.5169/seals-29458>

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Minimum Area of Circumscribed Polygons

1. Introduction

In [1] some estimates on minimal areas of polygons circumscribed about a plane convex set were considered. In what follows we shall prove a theorem that leads to very concise proofs of those estimates and some other results concerning circumscribed polygons.

We shall deal mainly with plane convex bodies. If K is a plane convex body, the area of K will usually be denoted by the same symbol K in order to simplify notation. We shall say that two convex n -gons are *parallel* if corresponding sides are parallel. Then we can state the main theorem as follows.

Theorem 1. Suppose K is a plane convex body, p is a polygon inscribed in K , and P is a polygon parallel to p and circumscribed about K . Then

$$K^2 \geq pP. \quad (1)$$

2. Proof of the main theorem

The proof of Theorem 1 depends on Minkowski's concept of the *mixed area*, $A(K, L)$, of two plane convex bodies K and L . In case p and P are parallel n -gons, $A(p, P)$ is easily described as follows. Let O be a point fixed interior to P . If l_i is the length of a side of p , let d_i be the distance from O to the corresponding parallel side of P . Then

$$A(p, P) = \frac{1}{2} \sum d_i l_i, \quad (2)$$

summed over all sides of p . In [5] one can find a treatment of the properties of mixed areas and a proof of the following fundamental inequality of Minkowski:

$$A(K, L)^2 \geq KL. \quad (3)$$

Now consider a plane convex body K , with inscribed n -gon p and parallel circumscribed n -gon P . Each side of P contains at least one point of K . If we choose one such point on each side of P , then these points, taken together with the vertices of p , are the vertices of a convex $2n$ -gon Q inscribed in K . Fix a point O inside p . If l_i is the length of a side of p , let d_i be the distance from O to the corresponding parallel side of P . Upon making a sketch of the situation, the reader will readily see that the area of Q is given by

$$Q = \frac{1}{2} \sum d_i l_i = A(p, P). \quad (4)$$

Using the fact that $Q \subset K$, and Minkowski's inequality, we then have,

$$K^2 \geq Q^2 = A(p, P)^2 \geq pP, \quad (5)$$

which proves Theorem 1.

3. Applications of the main theorem

We now derive a number of corollaries of Theorem 1, with all proofs following basically the same pattern.

Corollary 1. Any plane convex body K is contained in a triangle T_0 of area not more than twice that of K .

Proof. Let T_0 be a triangle of minimal area containing K . Then the midpoints of the sides of T_0 touch K (see [1] for a proof). Let t be the triangle inscribed in K formed by joining these midpoints, and let T be the triangle parallel to t and circumscribed about K . We have that $t = \frac{1}{4} T_0$ and $T \geq T_0$. Hence

$$K^2 \geq tT \geq \left(\frac{1}{4} T_0\right) (T_0) = \frac{1}{4} T_0^2, \quad (6)$$

so $T_0 \leq 2K$, as we wanted to prove.

Corollary 2. Any plane convex body K is contained in a quadrilateral Q_0 of area not more than $\sqrt{2}$ times that of K .

Proof. Let Q_0 be a quadrilateral of minimal area containing K . Again (see [1]) the midpoints of the sides of Q_0 touch K . Let q be the quadrilateral inscribed in K formed by joining the midpoints of the sides of Q_0 . Let Q be the quadrilateral parallel to q circumscribed about K . We have $Q \geq Q_0$, and it is easy to see q is a parallelogram with $q = \frac{1}{2} Q_0$. Hence

$$K^2 \geq qQ \geq \left(\frac{1}{2} Q_0\right) (Q_0) = \frac{1}{2} Q_0^2, \quad (7)$$

so $Q_0 \leq (\sqrt{2}) K$, as required.

The result given in Corollary 1 is in a sense the best possible, since a parallelogram K is not contained in any triangle of area less than twice the area of K . On the other hand, it is not known if the estimate for minimal circumscribed quadrilaterals in Corollary 2 is best possible, and good estimates for minimal circumscribed n -gons, $n > 4$, are apparently not known. However, the next corollary of Theorem 1 shows how to obtain an inequality by utilizing the maximum inscribed n -gon.

Corollary 3. Any plane convex body K is contained in an n -gon P of area not more than $\frac{2\pi}{n} \csc \frac{2\pi}{n}$ times that of K .

Proof. Let p be an n -gon of maximal area inscribed in K , and let P be the circumscribed n -gon parallel to p . By a theorem of Sas (see [4]), we have $p \geq \left(\frac{n}{2\pi} \sin \frac{2\pi}{n} \right) K$. Hence

$$K^2 \geq pP \geq \left(\frac{n}{2\pi} \sin \frac{2\pi}{n} \right) KP, \quad (8)$$

from which the result follows.

Suppose K is a centrally symmetric plane convex body. By a *lattice packing* of K we mean a distribution of translates of K , no pair having interior points in common, with their centers forming a plane lattice. The density of such a packing measures the fraction of the plane covered by these translates of K . The following result, proved in [4] in a different manner, follows readily from Theorem 1.

Corollary 4. Any centrally symmetric plane convex body K can be lattice packed with density at least $\frac{\sqrt{3}}{2}$.

Proof. By a theorem of Dowker [3], there is a centrally symmetric hexagon H_0 of minimum area circumscribed about K . A theorem of Day [2] implies that the mid-points of the sides of H_0 touch K . Let h be the hexagon formed by joining the mid-points of the sides of H_0 . Then it is not a difficult exercise to verify that h is the affine image of a regular hexagon, with $h = \frac{3}{4} H_0$. Let H be the centrally symmetric hexagon parallel to h and circumscribed about K . Then $H \geq H_0$, and

$$K^2 \geq hH \geq \left(\frac{3}{4} H_0 \right) (H_0) = \frac{3}{4} H_0^2, \quad (9)$$

so $K \geq \left(\frac{\sqrt{3}}{2} \right) H_0$. Since H_0 tiles the plane in a lattice manner, the required result follows.

4. Generalization to higher dimensions

Using mixed volumes in place of mixed areas, the following higher dimensional analogue of Theorem 1 is easily proved.

Theorem 2. Let K be a convex body in Euclidean n -space. Let p be a convex polytope contained in K and let P be a polytope circumscribed about K and parallel to p (that is, the facets of P parallel to corresponding facets of p). Then

$$K^n \geq p^{n-1} P, \quad (10)$$

where we are now using the same notational convention for volumes that we used before for areas.

Corollary 5. Any convex body K in Euclidean n -space is contained in a simplex T_0 of volume not more than n^{n-1} times that of K .

Proof. Let T_0 be a simplex of minimal volume containing K . By the theorem of Day [2], the centroids of the facets of T_0 touch K . Let t be the simplex whose vertices are those centroids, and let T be the simplex parallel to t and circumscribed about K . Then $t = (n^{-n}) T_0$ and $T \geq T_0$, so

$$K^n \geq t^{n-1} T \geq (n^{-n(n-1)} T_0^{n-1}) (T_0), \quad (11)$$

so $T_0 \leq (n^{n-1}) K$, as we wanted to prove.

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Hypo-Eulerian and Hypo-Traversable Graphs

Introduction

If a graph G does not possess a given property P , and for each vertex v of G the graph $G - v$ enjoys property P , then G is said to be a *hypo- P* graph. Recently, studies have been made where P stands for the graph being *hamiltonian*, *planar*, and *outerplanar* (e.g., see [3]). Here we obtain a characterization of *hypo-eulerian* and *hypo-randomly-eulerian* graphs, and investigate in this respect some of the other concepts arising out of Euler's solution of the classical Königsberg Seven Bridges Problem.

Preliminaries

Following the terminology of [2], a *graph* will be finite, undirected, without loops or multiple edges. A *walk* of a graph G is an alternating sequence $v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n$ of vertices and edges of G , beginning and ending with vertices and where the edge $e_i = v_{i-1} v_i$ for $i = 1, 2, \dots, n$. This is a $v_0 - v_n$ walk, and is usually denoted $v_0 v_1 v_2 \dots v_n$; it is *closed* if $v_0 = v_n$ and *open* otherwise. A walk is a *trail* if all its edges are distinct; it is a *path* if all its vertices are distinct. A closed trail is a circuit and a circuit on distinct vertices is a *cycle*. A cycle on p vertices is denoted C_p , and C_3 is called a *triangle*.

If for every two distinct vertices u and v of a graph G there exists a $u - v$ path, then G is *connected*. A *component* of G is a maximal connected subgraph of G . A vertex