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*Proof.* Let  $T_0$  be a simplex of minimal volume containing  $K$ . By the theorem of Day [2], the centroids of the facets of  $T_0$  touch  $K$ . Let  $t$  be the simplex whose vertices are those centroids, and let  $T$  be the simplex parallel to  $t$  and circumscribed about  $K$ . Then  $t = (n^{-n}) T_0$  and  $T \geq T_0$ , so

$$K^n \geq t^{n-1} T \geq (n^{-n(n-1)} T_0^{n-1}) (T_0), \quad (11)$$

so  $T_0 \leq (n^{n-1}) K$ , as we wanted to prove.

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## Hypo-Eulerian and Hypo-Traversable Graphs

### Introduction

If a graph  $G$  does not possess a given property  $P$ , and for each vertex  $v$  of  $G$  the graph  $G - v$  enjoys property  $P$ , then  $G$  is said to be a *hypo- $P$*  graph. Recently, studies have been made where  $P$  stands for the graph being *hamiltonian*, *planar*, and *outerplanar* (e.g., see [3]). Here we obtain a characterization of *hypo-eulerian* and *hypo-randomly-eulerian* graphs, and investigate in this respect some of the other concepts arising out of Euler's solution of the classical Königsberg Seven Bridges Problem.

### Preliminaries

Following the terminology of [2], a *graph* will be finite, undirected, without loops or multiple edges. A *walk* of a graph  $G$  is an alternating sequence  $v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n$  of vertices and edges of  $G$ , beginning and ending with vertices and where the edge  $e_i = v_{i-1} v_i$  for  $i = 1, 2, \dots, n$ . This is a  $v_0 - v_n$  walk, and is usually denoted  $v_0 v_1 v_2 \dots v_n$ ; it is *closed* if  $v_0 = v_n$  and *open* otherwise. A walk is a *trail* if all its edges are distinct; it is a *path* if all its vertices are distinct. A closed trail is a circuit and a circuit on distinct vertices is a *cycle*. A cycle on  $p$  vertices is denoted  $C_p$ , and  $C_3$  is called a *triangle*.

If for every two distinct vertices  $u$  and  $v$  of a graph  $G$  there exists a  $u - v$  path, then  $G$  is *connected*. A *component* of  $G$  is a maximal connected subgraph of  $G$ . A vertex

$v$  is a *cutpoint* of  $G$  if  $G - v$  has more components than  $G$ . An eulerian circuit of a graph  $G$  is a circuit which contains all the vertices and edges of  $G$ , and an *eulerian trail* of  $G$  is an open trail which contains all the vertices and edges of  $G$ ; in either case  $G$  has to be connected. We will assume that an eulerian circuit or an eulerian trail has at least one edge in it.

The number of edges incident with a vertex  $v$  is the *degree* of  $v$  which is written as  $\deg v$ . Let  $\delta(G) = \min_v \deg v$  and  $\Delta(G) = \max_v \deg v$ . A graph  $G$  is *regular of degree  $r$*  (or  *$r$ -regular*) if  $\delta(G) = \Delta(G) = r$ . A *cubic* graph is 3-regular. We use  $p(G)$  and  $q(G)$  (often simply  $p$  and  $q$ ) for the number of vertices and edges of a graph  $G$ . The *trivial* graph has  $p = 1$  and the *complete graph*  $K_p$  on  $p$  vertices has  $q = p(p-1)/2$ . The *complete bipartite graph*  $K(m, n)$  has its vertex set partitioned into nonempty sets  $V_1$  and  $V_2$  containing  $m$  and  $n$  elements respectively such that  $uv$  is an edge of  $K(m, n)$  if and only if  $u \in V_i$  and  $v \in V_j$ ,  $i \neq j$ .

An edge  $x = uv$  of a graph  $H$  is said to be *subdivided* if it is replaced by a new vertex  $w$  together with the edges  $uw$  and  $wv$ . A graph  $G$  is *homeomorphic from* a graph  $H$  if  $G$  can be obtained from  $H$  by a finite sequence of such subdivisions. Two graphs  $G_1$  and  $G_2$  are *homeomorphic* if there exists a graph  $G$  such that  $G_1$  and  $G_2$  are both homeomorphic from  $G$ .

Let  $\theta(G)$  ( $\xi(G)$ ) consist of the vertices of  $G$  having their degrees odd (even). Let the number of elements in  $\theta(G)$  be called the *euler number* of  $G$ , and let this be written as  $\epsilon(G)$ . Then  $\epsilon(G)$  is a nonnegative even integer.

## Hypo-eulerian Graphs

A graph  $G$  on  $p \geq 3$  vertices is defined to be *eulerian* if it possesses an eulerian circuit. The next result is well known.

*Theorem (Euler)*. Let  $G$  be a connected graph. Then  $G$  is eulerian if and only if  $\epsilon(G) = 0$ .

By definition, a graph  $G$  is *hypo-eulerian* if  $G$  is not eulerian, but the graph  $G - v$  is eulerian for each vertex  $v$  of  $G$ .

*Theorem 1*. Let  $G$  be a connected nontrivial graph. Then  $G$  is hypo-eulerian if and only if  $G = K_{2n}$ ,  $n \geq 2$ .

*Proof*. Clearly,  $\epsilon(K_{2n}) = 2n > 0$  and  $\epsilon(K_{2n} - v) = \epsilon(K_{2n-1}) = 0$  imply the sufficiency part. So let  $G$  be a nontrivial connected hypo-eulerian graph. As  $G - v$  is eulerian,  $p(G) \geq 4$ .

First we show that every vertex of  $G$  must be odd. Assume that  $\xi(G) \neq \emptyset$ , and let  $u \in \xi(G)$ . Now  $u$  must be adjacent with only odd vertices otherwise  $\epsilon(G - u) > 0$ . On the other hand if  $v \in \theta(G)$ , then for the same reason  $v$  must also be adjacent with only odd vertices. This contradicts  $\xi(G) \neq \emptyset$ . Hence  $p(G) = \epsilon(G) = 2n$  for some  $n \geq 2$ .

Secondly, we assert that  $G$  is complete. For if not, there exist two nonadjacent odd vertices  $u$  and  $v$  in  $G$ . Now the vertex  $v$  has odd degree in  $G - u$  and contradicts  $\epsilon(G - u) = 0$ . This completes the proof.

If  $G$  is an eulerian graph with  $p \geq 3$  and  $v$  is any vertex of  $G$ , then  $G - v$  necessarily contains odd vertices and must be noneulerian. This we mention next.

*Theorem 2.* Let  $G$  be a connected nontrivial graph. Then  $G$  is hypo-noneulerian if and only if  $G$  is eulerian.

Ore [4] called an eulerian graph  $G$  *randomly eulerian from a vertex  $v$*  if every trail of  $G$  beginning at  $v$  can be extended to an eulerian circuit of  $G$ ; a graph  $G$  is *randomly eulerian* if it is randomly eulerian from each of its vertices. Ore characterized graphs which are randomly eulerian from a vertex  $v$  as those graphs in which  $v$  belongs to every cycle of  $G$ . This leads to the result that  $G$  is randomly eulerian if and only if  $G$  is a cycle.

*Theorem 3.* A graph  $G$  is hypo-randomly-eulerian if and only if  $G = K_4$ .

*Proof.* Since a cycle is obtained by deleting any vertex of  $K_4$ , this graph certainly has the desired property. Conversely, let  $G$  be a hypo-randomly-eulerian graph. Observe that in view of Theorem 2,  $G$  and  $G - v$  cannot be both eulerian for any vertex  $v$ . Hence  $G$  is necessarily hypo-eulerian, and by Theorem 1,  $G = K_{2n}$  for some  $n \geq 2$ . Moreover, since  $G - v$  must be a cycle for each vertex  $v$  of  $K_{2n}$ , we conclude that  $G = K_4$ .

Chartrand and White [1] proved that if  $G$  is an eulerian graph which is randomly eulerian from  $k$  vertices, then  $k = 0, 1, 2$  or  $p(G)$ , and following this we will denote a graph which is randomly eulerian from  $k$  vertices as an  $RE(k)$  graph. A study of *hypo-RE(k)* graphs is now in order. Let  $G$  be a graph which is not  $RE(k)$ , but let  $G - v$  be randomly eulerian from  $k$  vertices. Then, as stated earlier,  $G$  must be a hypo-eulerian graph with the additional property that for all  $v$ ,  $G - v$  is an  $RE(k)$  graph. So by Theorem 1,  $G = K_{2n}$  and  $G - v = K_{2n-1}$ ,  $n \geq 2$ . When  $n \geq 3$ , for every vertex  $u$  of  $G - v$  we can find a cycle, namely a triangle, which avoids  $u$ , and so  $G - v$  is an  $RE(o)$  graph. The case  $n = 2$  yields that  $G - v$  is an  $RE(p)$  graph. Also,  $G - v$  is not an  $RE(k)$  graph for  $k = 1$  and  $k = 2$ . These remarks lead to the next result where we note that the *hypo-RE(p)* graphs have already been described in Theorem 3.

*Theorem 4.*

- (a) A graph  $G$  on  $p \geq 4$  vertices is *hypo-RE(o)* if and only if  $G = K_{2n}$ ,  $n \geq 3$ .
- (b) No graph is *hypo-RE(1)* or *hypo-RE(2)*.
- (c) A graph  $G$  on  $p \geq 4$  vertices is *hypo-RE(p)* if and only if  $G = K_4$ .

We conclude this section by stating a result analogous to Theorem 2.

*Theorem 5.* A graph  $G$  is *hypo-nonRE(k)* if and only if  $G$  is an  $RE(k)$  graph.

## Hypo-traversable Graphs

A graph  $G$  on  $p \geq 2$  vertices is said to be *traversable* if  $G$  has an eulerian trail, i. e.,  $G$  has an open trail which contains all the vertices and edges of  $G$  (and in view of the next result, this trail begins at one of the odd vertices and ends at the other).

*Theorem (Euler).* Let  $G$  be a connected graph. Then  $G$  is traversable if and only if  $\epsilon(G) = 2$ .

Let  $G$  be a hypo-traversable graph. Then  $\epsilon(G) \neq 2$ , and  $\epsilon(G - v) = 2$  for each vertex  $v$  of  $G$ . It is clear that  $G$  is a block, and  $\delta(G) \geq 2$ . Also,  $\epsilon(G)$  is even and  $0 \leq \epsilon(G) \leq p$ . From the first possible value we readily get the following.

*Theorem 6.* Let  $G$  be any connected graph which has euler number 0. Then  $G$  is hypo-traversable if and only if  $G$  is a cycle.

*Proof.* The sufficiency is immediate, and for the necessity we note that  $\epsilon(G) = 0$  implies that  $V(G) = \xi(G)$ . Now  $\epsilon(G - v) = 2$  for any vertex  $v$  of  $G$  gives  $\deg v = 2$ . By connectedness,  $G$  has to be a cycle.

Now let  $\epsilon(G) = 2m$ ,  $m \geq 2$ , and let  $G$  be hypo-traversable. Let  $u \in \xi(G)$  and  $v \in \theta(G)$ . Then it can be seen that  $\deg u = 2m - 2, 2m$  or  $2m + 2$  and  $\deg v = 2m - 3, 2m - 1$  or  $2m + 1$ , otherwise  $\epsilon(G - w) \neq 2$  for some vertex  $w$  of  $G$ . This fact is useful in considering individual cases. Should  $m = 2$ , the possible values of  $\deg v$  will be 3 or 5 since  $\delta(G) \geq 2$ . It can be verified that for  $p \leq 5$ , cycles are the only hypo-traversable graphs. Figure 1 shows all graphs on 6 vertices which are hypo-traversable.

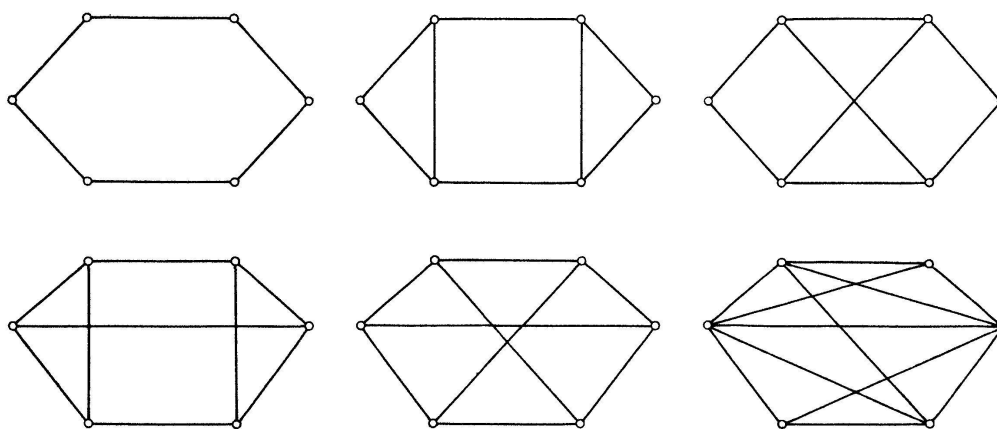


Figure 1

Hypo-traversable graphs on 6 vertices.

The preceding theorem dealt with the case when the graph had all vertices even. The next result treats graphs possessing no even vertices.

*Theorem 7.* Let  $G$  be any connected graph having euler number  $\epsilon(G) = p(G) \geq 6$ . Then  $G$  is hypo-traversable if and only if  $G$  is regular of degree  $p - 3$ .

*Proof.* Here  $\xi(G) = \phi$  and  $p = 2m = \epsilon(G)$ . By the above remarks, every vertex of  $G$  is odd and has possible degrees  $2m - 3$  or  $2m - 1$ . But if any vertex is adjacent with all the other  $p - 1$  vertices, its deletion gives an eulerian graph. The necessity now follows.

Conversely, let  $G$  be a connected  $(p - 3)$ -regular graph and  $\epsilon(G) = p(G) \geq 6$ . Then  $\epsilon(G - v) = 2$  for all  $v$ , and the proof is complete.

*Theorem 8.* Let  $G$  be a connected graph having euler number  $\epsilon(G) = p(G) - 1$ , and let  $p(G) \geq 5$ . Then  $G$  is hypo-traversable if and only if the even vertex  $u$  of  $G$  has degree  $p - 3$ , the vertices  $a$  and  $b$  that are nonadjacent with  $u$  have degree  $p - 4$ , and every other vertex has degree  $p - 2$ .

*Proof.* Let  $\xi(G) = \{u\}$ , and assume that  $G$  is hypo-traversable. Since every vertex adjacent with  $u$  becomes even in the traversable graph  $G - u$ , we need  $\deg u = p - 3$ . Let  $a$  and  $b$  be the vertices nonadjacent with  $u$ , and let  $v \in \theta(G) - \{a, b\}$ . Now the traversable graph  $G - w$  contains exactly 2 odd vertices, for each  $w \in V(G)$ .

Hence  $\deg v = p - 2$  and  $\deg a = \deg b = p - 4$ . For the sufficiency we note that  $\epsilon(G) \geq 4$ , and by hypothesis,  $\epsilon(G - w) = 2$  for each vertex  $w$  of  $G$ .

It is possible that a complete classification of hypo-traversable graphs may get involved with discussing individual cases, and this suggests scope for further research.

Let  $G$  be a hypo-nontraversable graph, i.e.,  $\epsilon(G) = 2$  and  $\epsilon(G - v) \neq 2$  for each vertex  $v$ . Moreover, since it is meaningful to require that  $G - v$  be connected, we further assume that  $G$  has no cutpoints and  $p \geq 4$  (so that  $\delta(G) \geq 2$ ). Designate the two odd vertices of  $G$  as  $a$  and  $b$ . If  $ab$  is not an edge in  $G$ , then  $\epsilon(G - a)$  and  $\epsilon(G - b)$  are 4 or more. On the other hand, if  $a$  and  $b$  are adjacent, we must have  $\deg a \geq 5$  and  $\deg b \geq 5$ . Now let  $v \in \xi(G)$ . This imposes the following restrictions: If  $\deg v = 2$ , then  $v$  is adjacent with either both or neither of  $a$  and  $b$ ; if  $\deg v = 4$ , then  $v$  is not simultaneously joined to both  $a$  and  $b$ . These present a set of necessary conditions for  $G$  to have the desired property, and it can be verified that they are also sufficient.

*Theorem 9.* Let  $G$  be a block with  $p \geq 4$ . Then  $G$  is hypo-nontraversable if and only if  $\theta(G) = \{a, b\}$  and

- (i)  $ab \in E(G) \Rightarrow \deg a \geq 5$  and  $\deg b \geq 5$ ,
- (ii)  $\deg v = 2 \Rightarrow v$  is joined to both or neither of  $a, b$ , and
- (iii)  $\deg v = 4 \Rightarrow v$  is not joined to both  $a$  and  $b$ .

In [1] a traversable graph  $G$  is called *randomly traversable from a vertex  $v$*  if every trail in  $G$  with initial vertex  $v$  can be extended to an eulerian trail of  $G$ . Clearly, a traversable graph can be randomly traversable from  $k = 0, 1$  or  $2$  vertices, and we may, as before, denote this class of graphs as  $RT(k)$ , where  $RT(2)$  will refer to the class of *randomly traversable graphs*. It was also proved in [1] that if  $a$  and  $b$  are the two odd vertices of a traversable graph  $G$ , then  $G$  is randomly traversable from  $a$  if and only if every cycle of  $G$  contains  $b$ . Moreover, a graph  $G$  is in  $RT(2)$  if and only if the two odd vertices of  $G$  lie on every cycle of  $G$ . This suggests the problem of studying *hypo-RT(k)* and *hypo-nonRT(k)* graphs.

We conclude by presenting a complete classification of  $RT(2)$  graphs.

*Theorem 10.* Let  $G$  be a traversable graph with  $\theta(G) = \{a, b\}$ . Then  $G$  is randomly traversable if and only if  $G$  is homeomorphic from  $K_2, K(2, 2m - 1)$  or  $K(2, 2m) + ab$ , where  $m \geq 1$ .

*Proof.* It is obvious that the graphs described are randomly traversable. To prove the converse, first we note that if  $\deg a = 1$ , then any  $b - a$  path must be  $G$  itself, otherwise there exists a cycle which avoids  $a$  or  $b$ . Thus,  $\deg b = 1$ , and the graph  $G$  is homeomorphic from  $K_2$ . So we assume that each of  $a$  and  $b$  has degree at least 3.

Let  $v$  be any vertex of  $G$  other than  $a$  or  $b$ . Since  $G$  is connected, there exist  $v - a$  and  $v - b$  paths. Clearly these paths have  $v$  as their only common vertex otherwise some cycle of  $G$  avoids  $a$  or  $b$ . Moreover, the union of these paths gives an  $a - b$  path which contains  $v$ . With every vertex  $v \in V(G) - \theta(G)$  we can associate an  $a - b$  path  $P(v)$  such that  $P(v)$  contains  $v$ . Let us consider the collection of all  $a - b$  paths, where, for obvious reasons, any two paths are disjoint, i.e., the only vertices common to them are  $a$  and  $b$ . So  $P(v)$  is unique, and the union of all these

paths must be  $G$  itself. We therefore conclude that every vertex other than  $a$  and  $b$  has degree 2, and  $\deg a = \deg b$  is odd. Also, if  $a$  and  $b$  are adjacent, then  $G - ab$  is homeomorphic from  $K(2, 2m)$ ; and if  $a, b$  are nonadjacent, then  $G$  is homeomorphic from  $K(2, 2m - 1)$ , where  $m \geq 1$ .

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## Kleine Mitteilungen

### New Quadratic Forms with High Density of Primes

Let  $p_{min}$  be the smallest prime contained in a quadratic form of the shape  $f(x) = Ax^2 + Ax - C$  and let  $n_{icp}$  be the number of initial consecutive primes of  $f(x)$ , then, by means of a CDC 6400 computer, all  $f(x) = Ax^2 + Ax - C$  were investigated for  $A < 10$ ,  $C < 2 \cdot 10^5$ , and  $p_{min} > 47$ . In Table 1, the number below  $C$  is the number of all primes of  $f(x)$  for  $x < 100$ , and  $p_{min}$  is the number in parentheses.

For each form  $x^2 + x - C$  we have also a form  $9y^2 + 9y - (C - 2)$ , because the substitution  $x = 3y + 1$  transforms  $x^2 + x - C$  into  $9y^2 + 9y - (C - 2)$ ; hence, each third term of  $x^2 + x - C$  (starting with the second) belongs to  $9y^2 + 9y - (C - 2)$ . Similarly, for each form  $2x^2 - C$  we have also a form  $8z^2 + 8z - (C - 2)$ , because the substitution  $x = 2z + 1$  transforms  $2x^2 - C$  into  $8z^2 + 8z - (C - 2)$ ; hence, each second term of  $2x^2 - C$  (starting with the second) belongs to  $8z^2 + 8z - (C - 2)$ . For the forms  $2x^2 - 119131$  and  $2x^2 - 186871$ , related to the forms with  $A = 8$  in Table 1, we have 64 and 61 primes, respectively, for  $x < 100$ .

Table 1 gives the impression that there might be no forms with  $A = 4$ . This is not so. In a test run with  $A < 10$ ,  $10^8 - 5000 < C < 10^8$ , and  $p_{min} > 47$ , the forms  $x^2 + x - 99995659$ ,  $9x^2 + 9x - 99995657$ , and  $4x^2 + 4x - 99996937$  were discovered, all with  $p_{min} = 53$ .

The form  $x^2 + x - 53509$  with  $p_{min} = 61$  is due to N. G. W. H. Beeger [1] in 1938, the forms  $x^2 + x - 90073$  with  $p_{min} = 53$  and  $x^2 + x - 169933$  with  $p_{min} = 59$  are due to the author [2] in 1967.

Two hundred years ago, Euler published his famous quadratic form  $x^2 + x + 41$  with  $p_{min} = 41$  and  $n_{icp} = 40$ . This form was believed to have the highest density of primes of all quadratic forms  $Ax^2 + Bx \pm C$  discovered till now. Many forms were found with  $p_{min} > 41$  and the second differences greater than 2; but the corresponding