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Hence $G_0 \neq G$, $G_{\{0,\infty\}} \neq G_0$. It is also clear that $(G_{\{0,\infty\}}, k-\{0\}, *)$ is transitive, for if $x \neq 0$ and $y \neq 0$, then

$$\begin{bmatrix} y & 0 \\ 0 & x \end{bmatrix} * x = y \, .$$

Hence by Theorem 3, $(G, X, *)$ is 3-fold transitive. We note that $(G, X, *)$ is not 4-fold transitive, for then $(G_{\{0,\infty\}}, k - \{0\}, *)$ would be 2-fold transitive.

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REFERENCE

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On 1-Factorability and Edge-Colorability of Cartesian Products of Graphs

There is no characterization of 1-factorable graphs. Thus, it is natural that many of the results on this topic have been the determination of classes of 1-factorable graphs. The object of this paper is to present a sufficient condition for the 1-factorability of the cartesian product of two graphs. We begin with some notation and definitions.

The vertex set of a graph G will be denoted by $V(G)$ and its edge set by $E(G)$. In this paper we consider only finite, undirected graphs without loops or multiple edges. Let G and H be two nonempty graphs for which $V(G) = V(H)$ and $E(G) \cap E(H) = \Phi$; then the graph G' is the *sum* of G and H, written $G' = G + H$, if $V(G') = V(G)$ and $E(G') = E(G) \cup E(H)$. A *1-factor* of a graph G is a spanning 1-regular subgraph of G. A graph is *1-factorable* if it can be expressed as a sum of edge-disjoint 1-factors. The *cartesian product* (or *product*) of the graph G with the graph H, denoted by $G \times H$, is defined by: $V(G \times H) = V(G) \times V(H)$; $E(G \times H) = \{[(u_1, v_1), (u_2, v_2)] \mid u_1 = u_2 \text{ and } v_1 v_2 \in E(H), \text{ or } v_1 = v_2 \text{ and } u_1 u_2 \in E(G)\}$.

An assignment of n colors to the edges of a nonempty graph G so that adjacent edges are colored differently is an *n-edge-coloring* of G. The minimum n for which a graph G is n-edge-colorable is its *edge-chromatic number* $\chi_1(G)$. By a theorem of Vizing [2], the edge-chromatic number $\chi_1(G)$ of a graph G is bounded by: $\Delta(G) \leq \chi_1(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. If G is regular, then G is 1-factorable if and only if $\chi_1(G) = \Delta(G)$. Hence any theorem concerning the 1-factorability of regular graphs has as an immediate corollary a result concerning edge-colorability, which is useful since there is also no characterization of those graphs which are $\Delta(G)$-edge-colorable. For other notations and definitions, we follow [1].

If K_2 denotes the complete graph on two vertices, then $K_2 \times H$, where H is any regular graph, is shown to be 1-factorable in the following lemma.

Lemma: If H is a regular graph, then $K_2 \times H$ is 1-factorable.

Proof. If H is 1-factorable, then the result follows immediately. Hence we consider the case that H is not 1-factorable. If H is an r-regular graph, then by a previous remark, $\chi_1(H) = r+1$. Let an $(r+1)$-edge-coloring of H be given and let $C_1, C_2, \ldots, C_{r+1}$ be the edge-color classes of $E(H)$. Now $K_2 \times H$ contains two disjoint copies of H. Let the $(r+1)$-edge-coloring of H be applied to these disjoint copies, and assign to each edge $[(u_1, v), (u_2, v)]$ of $K_2 \times H$ the only color among the $r+1$ colors which was assigned to no edge of H incident with v. Hence $K_2 \times H$ may be $(r+1)$-edge-colored. But $K_2 \times H$ is $(r+1)$-regular. Hence $\chi_1(K_2 \times H) = r+1$, and $K_2 \times H$ is 1-factorable.

We now state and prove the main result.

Theorem: If G is a 1-factorable graph and H is a regular graph, then $G \times H$ is a 1-factorable graph.

Proof: Let G be a 1-factorable, r-regular graph of order p_1 with 1-factors $G_1, G_2, \ldots, G_r$, and let H be an s-regular graph of order p_2. Then consider the subgraphs $G_1 \times H, G_2 \times \overline{K_{p_2}}, \ldots, G_r \times \overline{K_{p_2}}$ of $G \times H$, where $\overline{K_{p_2}}$ denotes the graph consisting of p_2 isolated vertices. Note that these subgraphs are mutually edge-disjoint subgraphs spanning $G \times H$, and $G \times H = G_1 \times H + \sum_{i=2}^{r} G_i \times \overline{K_{p_2}}$. Moreover, the subgraphs $G_2 \times \overline{K_{p_2}}, \ldots, G_r \times \overline{K_{p_2}}$ are 1-regular and thus are 1-factors of $G \times H$. Hence if $G_1 \times H$ is 1-factorable, $G \times H$ is 1-factorable. Now $G_1 \times H$ is a spanning $(s+1)$-regular subgraph of $G \times H$ consisting of $p_1/2$ components each of which is isomorphic to $K_2 \times H$. By the Lemma, $K_2 \times H$ is 1-factorable and of regularity $s+1$. Let the 1-factors of $K_2 \times H$ be $F_1, F_2, \ldots, F_{s+1}$ in a 1-factorization of $K_2 \times H$. Select in every component of $G_1 \times H$, the same 1-factor F_k, where $1 \leq k \leq s+1$, and designate the resultant subgraph of $G_1 \times H$ by F_k'. Then by the choice of F_k' it follows that F_k' is a spanning 1-regular subgraph of $G_1 \times H$, and hence a 1-factor of $G_1 \times H$. In a like manner mutually edge-disjoint 1-factors $F_1', F_2', \ldots, F_{s+1}'$ of $G_1 \times H$ can be obtained from each of $F_1, F_2, \ldots, F_{s+1}$, respectively. Therefore $G_1 \times H$ is 1-factorable, which implies that $G \times H$ is also 1-factorable as previously indicated.

Corollary: If G and H are regular graphs, and $\chi_1(G) = \Delta(G)$, then $\chi_1(G \times H) = \Delta(G) + \Delta(H)$.

We remark that the theorem gives a sufficient condition for 1-factorability which is, however, not a necessary condition, since 1-factorable products of two non-1-factorable graphs are known. An example of this is the cartesian product of the Petersen graph with a triangle.

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