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Kleine Mitteilungen

Note on defining arc length

We prove here that for piecewise convex (plane) curves there exist simple and intuitive *axioms specifying arc length* precisely. Our axioms (and our proof) could readily be included in a freshman or sophomore calculus course. First let us review some approaches to arc length.

(1) One may define a curve Γ to be *rectifiable* if $L =$ (the sup of lengths of polygons inscribed in this curve) exists $< \infty$. A typical theorem is that if the functions in some parametric representation have continuous derivatives, then $L < \infty$ and L is calculable by the familiar integral. (See e.g. [2, §79] or [4, pp. 13–14].) In this approach L is called the length of Γ without further question.

(2) More modern treatments have a slightly more skeptical approach. For example in Apostol [1, p. 247], the need to give an intuitively (and mathematically) acceptable definition of arc length is more clearly felt than in the earlier works mentioned above. Nonetheless the assertion there that clearly the length of Γ must be defined as $\geq L$ (above) is certainly more compelling than the subsequent weaker assertion that "it seems reasonable to define the length of the curve to be [the above number L]".

(3) There exists a completely axiomatic approach involving Lebesgue measure and integration, in arbitrary finite dimension [5, Chapter 9]. But one of the axioms there is a disguised version of the ordinary formula for arc length.

(4) J. Mycielski kindly mentioned to the author the work of Zykov [6], who also pointed out the need for an upper bound on arc length. His treatment is more general than ours, but less axiomatic and less accessible to calculus students.

We now state our axioms for arc length, which apply (at least) to the class $C^1(R)$ of functions having continuous derivative. In the axioms, $L(f, a, b)$ denotes the length of the curve defined by f on the interval $[a, b]$.

Axioms

Ax. 1. Two arcs which are congruent (i.e. under rigid motions) have the same length.

Ax. 2. If $a \leq b \leq c$ and $f \in C^1$, then $L(f, a, c) = L(f, a, b) + L(f, b, c)$.

Ax. 3. If f is a constant function, then $L(f, a, b) = b - a$.

Ax. 4. (Shortest distance) If $\bar{f}$ is a linear function such that $\bar{f}(a) = f(a)$ and $\bar{f}(b) = f(b)$, then $L(\bar{f}, a, b) \leq L(f, a, b)$.

Ax. 5. If $| f'(t) | \leq | g'(t) |$ for $a \leq t \leq b$, then $L(f, a, b) \leq L(g, a, b)$.

Let us say that $f \in C^1(R)$ is *convex* on the interval $[a, b]$ iff $f'(t)$ is a monotone (increasing or decreasing) function of t for $a \leq t \leq b$, and that f is *piecewise convex* iff there exist numbers $a_1 < \ldots < a_N$ such that f is convex on each interval $[-\infty, a_1]$, $[a_1, a_2], \ldots, [a_{N-1}, a_N]$, $[a_N, \infty]$.

Theorem. There exists an arc-length function $L(f, a, b)$, defined for all $f \in C^1$, obeying Axioms 1–5. Such L is unique on piecewise convex f.

Proof. It is well known that

$$L(f, a, b) = \int_a^b \sqrt{1 + (f')^2} \tag{1}$$

satisfies the axioms. Thus we need only show that if $L(\cdot, \cdot, \cdot)$ obeys the axioms, then (1) holds for every piecewise convex f. By Axioms 1 and 2, we may assume that f' is monotone increasing on $[a, b]$. Furthermore, Axioms 1, 2 and 3 imply that segments must have their usual length. Proceeding in the familiar way (see e.g. [3, p. 270]) we define

$$S(x) = L(f, a, x) \qquad (a \leq x \leq b),$$

and calculate that

$$\frac{1}{h}\,[S\,(x+h) - S(x)] = \sqrt{1+[f'(\xi)]^2}\,\left\{\frac{L\,(f,\,x,\,x+h)}{L\,(\bar{f},\,x,\,x+h)}\right\}, \tag{2}$$

where $x < \xi < x + h$ and $\bar{f}$ is a linear function such that $\bar{f}(x) = f(x)$ and $\bar{f}(x+h) = f(x+h)$. It follows immediately from the next lemma and Ax. 4 that for fixed x the limit of the expression $\{\cdot\}$ in (2) is 1 as h tends to 0, thus establishing that

$$S'(x) = \sqrt{1 + [f'(x)]^2},$$

which in turn implies (1). Q.E.D.

Lemma. Assume Axioms 1–5; let $f \in C^1(R)$ be convex on $[a, b]$, let P be $(a, f(a))$, R be $(b, f(b))$ and let Q be the point of intersection of the tangent lines to the curve $y = f(x)$ at P and at R. Then $L(f, a, b) \leq PQ + QR$ (the sum of two distances).

Proof. By convexity there exists a unique point Q' nearest to Q on the arc of the curve $y = f(x)$ $(a \leq x \leq b)$. Clearly by Axiom 1 we may assume that Q is directly below Q' (= say (c, d)). By Axiom 5, we see that $L(f, a, c) \leq PQ$ and $L(f, c, b) \leq QR$. Clearly Axiom 2 completes the proof of the Lemma.

J. Mycielski has remarked that more advanced techniques will probably allow one to apply the axiom to a wider class of functions.

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The Generating Function for $\{\min(n_1, \ldots, n_k)\}^m$

In [1], L. Carlitz has evaluated certain series including

$$G = \sum_{n_1,\ldots,n_k=0}^{\infty} \{\min(n_1, \ldots, n_k)\}^m x_1^{n_1} \ldots x_k^{n_k}$$

for $k=3$ and $m=2,3$. In this note we evaluate G where k and m are arbitrary positive integers. The derivation is quite simple once we observe that

$$\{\min(n_1,\ldots,n_k)\}^m = \sum_{j=0}^{\min(n_1,\ldots,n_k)} j^m - \sum_{j=0}^{\min(n_1,\ldots,n_k)-1} j^m$$
$$= \sum_{j+i_1=n_1,\ldots,j+i_k=n_k} j^m - \sum_{j+i_1+1=n_1,\ldots,j+i_k+1=n_k} j^m .$$

Thus

$$G = \sum_{n_1,\ldots,n_k=0}^{\infty} \left\{ \sum_{j+i_1=n_1,\ldots,j+i_k=n_k} j^m x_1^{n_1}\ldots x_k^{n_k} - \sum_{j+i_1+1=n_1,\ldots,j+i_k+1=n_k} j^m x_1^{n_1}\ldots x_k^{n_k} \right\}$$
$$= \sum_{i_1,\ldots,i_k,j=0}^{\infty} j^m x_1^{j+i_1}\ldots x_k^{j+i_k} - \sum_{i_1,\ldots,i_k,j=0}^{\infty} j^m x_1^{j+i_1+1}\ldots x_k^{j+i_k+1}$$
$$= (1-x_1x_2\ldots x_k)(1-x_1)^{-1}\ldots(1-x_k)^{-1}\sum_{j=0}^{\infty} j^m (x_1x_2\ldots x_k)^j$$
$$= \frac{a_m(x_1x_2\cdots x_k)}{(1-x_1)\cdots(1-x_k)(1-x_1x_2\cdots x_k)^m}$$

where $a_m(y)$ is the m^{th} Eulerian polynomial (see [2] p. 38).

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Aufgaben

Aufgabe 693. Es seien A ein offenes Intervall der reellen Zahlengeraden und f, g zwei auf A definierte differenzierbare reellwertige Funktionen mit $f(x) \neq \pm 1$, $f(x) \neq 0$, $g(x) \neq 0$ für alle x in A. Man zeige, dass die durch

$$F(x) = {}^{|f(x)|}\log|g(x)| \quad [x \in A]$$

definierte Funktion F differenzierbar ist und bestimme die erste Ableitung von F.

R. Rose, Biel

Lösung: Es ist

$$F(x) = \frac{\ln|g(x)|}{\ln|f(x)|} \quad [x \in A].$$

Hier sind nach Voraussetzung und der Kettenregel Zähler- und Nennerfunktion für alle $x \in A$ differenzierbar, und es gilt z.B.

$$(\ln|g(x)|)' = \begin{cases} (\ln g(x))' = \dfrac{g'(x)}{g(x)} & \text{wenn } g(x) > 0 \\[2ex] (\ln(-g(x)))' = \dfrac{-g'(x)}{-g(x)} = \dfrac{g'(x)}{g(x)} & \text{wenn } g(x) < 0. \end{cases}$$