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A note on Bernoulli numbers and polynomials

Put

$$S_k = S_k(n) = \sum_{a=0}^{n-1} a^k .$$

It is well known that

$$S_1^2 = S_3, \; 2\, S_1^4 = S_5 + S_7 . \tag{1}$$

The general formula of this type was found by Stern [1, p. 20]:

$$2^{m-1}\, S_1^m = \sum_{2j<m} \binom{m}{2j+1} S_{2m-2j-1} . \tag{2}$$

We recall that [2, Ch. 2]

$$S_k(n) = \frac{B_{k+1}(n) - B_{k+1}}{k+1}, \tag{3}$$

where $B_{k+1}(n)$ is the Bernoulli polynomial of degree $k+1$ and $B_{k+1} = B_{k+1}(0)$. The Bernoulli polynomial may be defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!} . \tag{4}$$

Substituting from (3) in (2) we get

$$2^{m-1}\, S_1^m = \sum_{2j<m} \binom{m}{2j+1} \frac{B_{2m-2j}(n) - B_{2m-2j}}{2m-2j} . \tag{5}$$

Since $S_1 = n\,(n-1)/2$, it is clear that (5) is a polynomial identity for $n = 1, 2, 3, \ldots$ Therefore we may write

$$(x\,(x-1))^m = 2 \sum_{2j<m} \binom{m}{2j+1} \frac{B_{2m-2j}(x) - B_{2m-2j}}{2m-2j} .$$

Since $B'_m(x) = mB_{m-1}(x)$, it follows that

$$m\,(2x-1)\;(x\,(x-1))^{m-1} = 2 \sum_{2j<m} \binom{m}{2j+1} B_{2m-2j-1}(x) . \tag{6}$$

Conversely integration of (6) gives (5).

A slightly simpler formula is

$$(m+1)\;(x - {}^1\!/_2)^m = \sum_{2j\le m} \binom{m+1}{2j+1} 2^{-2j} B_{m-2j}(x) . \tag{7}$$

To prove (7) we make use of (4). Clearly

$$\sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!} = \frac{ze^{(x-1/2)z}}{e^{z/2} - e^{-z/2}},$$

so that

$$ze^{(x-1/2)z} = 2 \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!} \sum_{j=0}^{\infty} \frac{(z/2)^{2j+1}}{(2j+1)!}.$$

Equating coefficients of z^{m+1}, we get (7).

Next we recall the expansion [2, p. 28]

$$B_k(x) = \sum_{s=0}^{k} \binom{k}{s} 2^{-s} D_s (x - {}^1/_2)^{k-s} \tag{8}$$

where

$$D_k = 2^k B_k({}^1/_2) = 2(1 - 2^{k-1}) B_k. \tag{9}$$

It follows from (9) that

$$D_{2k+1} = 0, \ (-1)^k D_{2k} > 0 \qquad (k = 0, 1, 2, \ldots). \tag{10}$$

The first few values are

$$D_0 = 1, \ D_2 = -\frac{1}{3}, \ D_4 = \frac{7}{15}, \ D_6 = -\frac{31}{21}.$$

Thus (7) and (8) are an inverse pair. It is convenient to consider separately even and odd values of m and k. Replacing m by $2m$, (7) becomes

$$(2m+1)\ (x - {}^1/_2)^{2m} = \sum_{j=0}^{m} \binom{2m+1}{2j} 2^{-2m+2j} B_{2j}(x). \tag{11}$$

Similarly, by (8) and (10)

$$B_{2k}(x) = \sum_{s=0}^{k} \binom{2k}{2s} 2^{-2k+2s} D_{2k-2s} (x - {}^1/_2)^{2s}. \tag{12}$$

Substituting from (12) in (11), we get

$$(2m+1)\ (x - {}^1/_2)^{2m} = \sum_{j=0}^{m} \binom{2m+1}{2j} 2^{-2m} \sum_{s=0}^{j} \binom{2j}{2s} 2^{2s} D_{2j-2s} (x - {}^1/_2)^{2s},$$

so that

$$\sum_{j=s}^{m} \binom{2m+1}{2j} \binom{2j}{2s} 2^{2s} D_{2j-2s} = (2m+1)\ 2^{2m} \delta_{m,s}. \tag{13}$$

For odd values of the parameters, (7) and (8) yield

$$(2m+2)\ (x - {}^1/_2)^{2m+1} = \sum_{j=0}^{m} \binom{2m+2}{2j+1} 2^{-2m+2j} B_{2j+1}(x), \tag{14}$$

and

$$B_{2k+1}(x) = \sum_{s=0}^{k} \binom{2k+1}{2s+1} 2^{-2k+2s} D_{2k-2s} (x - {}^1/_2)^{2s+1}, \tag{15}$$

respectively. Hence

$$(2m+2)\ (x - {}^1/_2)^{2m+1} = \sum_{j=0}^{m} \binom{2m+2}{2j+1} 2^{-2m} \sum_{s=0}^{j} \binom{2j+1}{2s+1} 2^{2s} D_{2j-2s} (x - {}^1/_2)^{2s+1},$$

so that

$$\sum_{j=s}^{m} \binom{2m+2}{2j+1} \binom{2j+1}{2s+1} 2^{2s} D_{2j-2s} = (2m+2)\, 2^{2m}\, \delta_{m,s}\,. \tag{16}$$

The formulas (13), (16) evidently imply the following matrix formulas:

$$\left[\binom{2m+1}{2s}\right] \left[\binom{2m}{2s} 2^{2s} D_{2m-2s}\right] = \left[(2m+1)\, 2^{2m}\, \delta_{m,s}\right], \tag{17}$$

$$\left[\binom{2m+2}{2s+1}\right] \left[\binom{2m+1}{2s+1} 2^{2s} D_{2m-2s}\right] = \left[(2m+2)\, 2^{2m}\, \delta_{m,s}\right], \tag{18}$$

where $m, s = 0, 1, 2, \ldots, N-1$ and N is either a positive integer or infinity. Thus in particular we have found inverses of the matrices

$$\left[\binom{2m+1}{2s}\right], \quad \left[\binom{2m+2}{2s+1}\right] \qquad (m, s = 0, 1, 2, \ldots, N-1)\,. \tag{19}$$

For applications it is convenient to state the following result.

Theorem 1. *The set of equations*

$$(m+1)\, x_m = \sum_{2j \leq m} \binom{m+1}{2j+1} 2^{-2j}\, y_{m-2j} \qquad (m = 0, 1, 2, \ldots) \tag{20}$$

is equivalent to the set

$$y_m = \sum_{2j \leq m} \binom{m}{2j} 2^{-2j} D_{2j}\, x_{m-2j} \qquad (m = 0, 1, 2, \ldots). \tag{21}$$

Separating even and odd values of m we have

Theorem 2. *The set of equations*

$$(2m+1)\, x_m = \sum_{j=0}^{m} \binom{2m+1}{2j} 2^{2j-2m}\, y_j \qquad (m = 0, 1, 2, \ldots) \tag{22}$$

is equivalent to the set

$$y_m = \sum_{j=0}^{m} \binom{2m}{2j} 2^{2j-2m} D_{2m-2j}\, x_j \qquad (m = 0, 1, 2, \ldots)\,. \tag{23}$$

The set of equations

$$(2m+2)\, x_m = \sum_{j=0}^{m} \binom{2m+2}{2j+1} 2^{2j-2m}\, y_j \qquad (m = 0, 1, 2, \ldots) \tag{24}$$

is equivalent to the set

$$y_m = \sum_{j=0}^{m} \binom{2m+1}{2j+1} 2^{2j-2m} D_{2m-2j}\, x_j \qquad (m = 0, 1, 2, \ldots)\,. \tag{25}$$

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