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## A note on Bernoulli numbers and polynomials

Put

$$S_k = S_k(n) = \sum_{a=0}^{n-1} a^k.$$

It is well known that

$$S_1^2 = S_3, \quad 2 S_1^4 = S_5 + S_7. \quad (1)$$

The general formula of this type was found by Stern [1, p. 20]:

$$2^{m-1} S_1^m = \sum_{2j < m} \binom{m}{2j+1} S_{2m-2j-1}. \quad (2)$$

We recall that [2, Ch. 2]

$$S_k(n) = \frac{B_{k+1}(n) - B_{k+1}}{k+1}, \quad (3)$$

where  $B_{k+1}(n)$  is the Bernoulli polynomial of degree  $k+1$  and  $B_{k+1} = B_{k+1}(0)$ . The Bernoulli polynomial may be defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!}. \quad (4)$$

Substituting from (3) in (2) we get

$$2^{m-1} S_1^m = \sum_{2j < m} \binom{m}{2j+1} \frac{B_{2m-2j}(n) - B_{2m-2j}}{2m-2j}. \quad (5)$$

Since  $S_1 = n(n-1)/2$ , it is clear that (5) is a polynomial identity for  $n = 1, 2, 3, \dots$ . Therefore we may write

$$(x(x-1))^m = 2 \sum_{2j < m} \binom{m}{2j+1} \frac{B_{2m-2j}(x) - B_{2m-2j}}{2m-2j}.$$

Since  $B'_m(x) = mB_{m-1}(x)$ , it follows that

$$m(2x-1)(x(x-1))^{m-1} = 2 \sum_{2j < m} \binom{m}{2j+1} B_{2m-2j-1}(x). \quad (6)$$

Conversely integration of (6) gives (5).

A slightly simpler formula is

$$(m+1)(x-1/2)^m = \sum_{2j \leq m} \binom{m+1}{2j+1} 2^{-2j} B_{m-2j}(x). \quad (7)$$

To prove (7) we make use of (4). Clearly

$$\sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!} = \frac{ze^{(x-1/2)z}}{e^{z/2} - e^{-z/2}},$$

so that

$$ze^{(x-1/2)z} = 2 \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!} \sum_{j=0}^{\infty} \frac{(z/2)^{2j+1}}{(2j+1)!}.$$

Equating coefficients of  $z^{m+1}$ , we get (7).

Next we recall the expansion [2, p. 28]

$$B_k(x) = \sum_{s=0}^k \binom{k}{s} 2^{-s} D_s (x - 1/2)^{k-s} \quad (8)$$

where

$$D_k = 2^k B_k(1/2) = 2(1 - 2^{k-1}) B_k. \quad (9)$$

It follows from (9) that

$$D_{2k+1} = 0, \quad (-1)^k D_{2k} > 0 \quad (k = 0, 1, 2, \dots). \quad (10)$$

The first few values are

$$D_0 = 1, \quad D_2 = -\frac{1}{3}, \quad D_4 = \frac{7}{15}, \quad D_6 = -\frac{31}{21}.$$

Thus (7) and (8) are an inverse pair. It is convenient to consider separately even and odd values of  $m$  and  $k$ . Replacing  $m$  by  $2m$ , (7) becomes

$$(2m+1) (x - 1/2)^{2m} = \sum_{j=0}^m \binom{2m+1}{2j} 2^{-2m+2j} B_{2j}(x). \quad (11)$$

Similarly, by (8) and (10)

$$B_{2k}(x) = \sum_{s=0}^k \binom{2k}{2s} 2^{-2k+2s} D_{2k-2s} (x - 1/2)^{2s}. \quad (12)$$

Substituting from (12) in (11), we get

$$(2m+1) (x - 1/2)^{2m} = \sum_{j=0}^m \binom{2m+1}{2j} 2^{-2m} \sum_{s=0}^j \binom{2j}{2s} 2^{2s} D_{2j-2s} (x - 1/2)^{2s},$$

so that

$$\sum_{j=s}^m \binom{2m+1}{2j} \binom{2j}{2s} 2^{2s} D_{2j-2s} = (2m+1) 2^{2m} \delta_{m,s}. \quad (13)$$

For odd values of the parameters, (7) and (8) yield

$$(2m+2) (x - 1/2)^{2m+1} = \sum_{j=0}^m \binom{2m+2}{2j+1} 2^{-2m+2j} B_{2j+1}(x), \quad (14)$$

and

$$B_{2k+1}(x) = \sum_{s=0}^k \binom{2k+1}{2s+1} 2^{-2k+2s} D_{2k-2s} (x - 1/2)^{2s+1}, \quad (15)$$

respectively. Hence

$$(2m+2) (x - 1/2)^{2m+1} = \sum_{j=0}^m \binom{2m+2}{2j+1} 2^{-2m} \sum_{s=0}^j \binom{2j+1}{2s+1} 2^{2s} D_{2j-2s} (x - 1/2)^{2s+1},$$

so that

$$\sum_{j=s}^m \binom{2m+2}{2j+1} \binom{2j+1}{2s+1} 2^{2s} D_{2j-2s} = (2m+2) 2^{2m} \delta_{m,s}. \quad (16)$$

The formulas (13), (16) evidently imply the following matrix formulas:

$$\left[ \binom{2m+1}{2s} \right] \left[ \binom{2m}{2s} 2^{2s} D_{2m-2s} \right] = \left[ (2m+1) 2^{2m} \delta_{m,s} \right], \quad (17)$$

$$\left[ \binom{2m+2}{2s+1} \right] \left[ \binom{2m+1}{2s+1} 2^{2s} D_{2m-2s} \right] = \left[ (2m+2) 2^{2m} \delta_{m,s} \right], \quad (18)$$

where  $m, s = 0, 1, 2, \dots, N-1$  and  $N$  is either a positive integer or infinity. Thus in particular we have found inverses of the matrices

$$\left[ \binom{2m+1}{2s} \right], \quad \left[ \binom{2m+2}{2s+1} \right] \quad (m, s = 0, 1, 2, \dots, N-1). \quad (19)$$

For applications it is convenient to state the following result.

**Theorem 1.** *The set of equations*

$$(m+1) x_m = \sum_{2j \leq m} \binom{m+1}{2j+1} 2^{-2j} y_{m-2j} \quad (m = 0, 1, 2, \dots) \quad (20)$$

is equivalent to the set

$$y_m = \sum_{2j \leq m} \binom{m}{2j} 2^{-2j} D_{2j} x_{m-2j} \quad (m = 0, 1, 2, \dots). \quad (21)$$

Separating even and odd values of  $m$  we have

**Theorem 2.** *The set of equations*

$$(2m+1) x_m = \sum_{j=0}^m \binom{2m+1}{2j} 2^{2j-2m} y_j \quad (m = 0, 1, 2, \dots) \quad (22)$$

is equivalent to the set

$$y_m = \sum_{j=0}^m \binom{2m}{2j} 2^{2j-2m} D_{2m-2j} x_j \quad (m = 0, 1, 2, \dots). \quad (23)$$

The set of equations

$$(2m+2) x_m = \sum_{j=0}^m \binom{2m+2}{2j+1} 2^{2j-2m} y_j \quad (m = 0, 1, 2, \dots) \quad (24)$$

is equivalent to the set

$$y_m = \sum_{j=0}^m \binom{2m+1}{2j+1} 2^{2j-2m} D_{2m-2j} x_j \quad (m = 0, 1, 2, \dots). \quad (25)$$

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