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QR in Two Dimensions

The *QR* process is one of the most efficient ways of determining the characteristic values of a matrix. It is a unitary analog of the *LR* process of RUTISHAUSER [1]. However even the best proofs available are unfit for beginners' consumption and the later developments of the process are not yet fully understood. We present here a discussion of the two-dimensional case, in its simplest form. The formal description of the process will be given in the n-dimensional case.

Let A be a complex $n \times n$ matrix. It is well-known that A can be written in the form

$$A = QR \tag{1}$$

where Q is unitary and R upper triangular. This is essentially the result of the Gram-Schmidt orthogonalization process. Moreover, if we require the diagonal elements of R to be positive, then the representation (1) is unique.

The QR-algorithm consists in deriving sequences of matrices $\{A_n\}$, $\{Q_n\}$, $\{R_n\}$ from $A = A_1$ by repeated use of (1). Given $A_n = Q_n R_n$ we form the reversed product $A_{n+1} = R_n Q_n$ and factorize this as $A_{n+1} = Q_{n+1} R_{n+1}$. Since

$$A_{n+1} = R_n Q_n = (Q_n^* Q_n) R_n Q_n = Q_n^* (Q_n R_n) Q_n = Q_n^* A_n Q_n \tag{2}$$

the matrices $\{A_n\}$ are all (unitarily) similar to A and have the same characteristic values as A. The basic fact is that, in certain circumstances, the sequence $\{A_n\}$ converges geometrically to an upper triangular matrix, which has the characteristic values of A on the diagonal. For discussions of this result see the original papers of H. RUTISHAUSER [1], J.G.F. FRANCIS [2], V.N. KUBLANOWSKAJA [3] and more recent work of B. PARLETT [4, 5, 6], A.S. HOUSEHOLDER [7], G.W. STEWART [8] and J.H. WILKINSON [9].

In practice, appropriate "shifts" are introduced and quadratic convergence can be obtained.

For simplicity we discuss the real two dimensional case. Let

$$A=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$$

be real and unimodular, det $A=1$. We compute the QR decomposition. If, where $c=\cos\theta$, $s=\sin\theta$,

$$A=A_1=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}=\begin{bmatrix}c & s\\ s & -c\end{bmatrix}\begin{bmatrix}\xi & \eta\\ 0 & \zeta\end{bmatrix}$$

then

$$c=\alpha\,(\alpha^2+\gamma^2)^{-1/2},\ s=\gamma\,(\alpha^2+\gamma^2)^{-1/2}.$$

Next

$$A_2=\begin{bmatrix}c & s\\ s & -c\end{bmatrix}\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\begin{bmatrix}c & s\\ s & -c\end{bmatrix}=\begin{bmatrix}(\alpha+\delta-(\alpha/(\alpha^2+\gamma^2)) & \gamma-\beta-(\gamma/(\alpha^2+\gamma^2))\\ -\gamma/(\alpha^2+\gamma^2) & \alpha/(\alpha^2+\gamma^2)\end{bmatrix}.$$

If we write

$$A_n=\begin{bmatrix}\alpha_n & \beta_n\\ \gamma_n & \delta_n\end{bmatrix}$$

then the recurrence relations determining α_{n+1}, γ_{n+1} are

$$\begin{cases}\alpha_{n+1}=(\alpha_1+\delta_1)-(\alpha_n/(\alpha_n^2+\gamma_n^2)),\\ \gamma_{n+1}=-\gamma_n/(\alpha_n^2+\gamma_n^2),\end{cases}\qquad n=1,2,\ldots \tag{3}$$

What we have to prove from (3) is that, in certain circumstances

$$\alpha_n\to\lambda,\ \gamma_n\to 0$$

where λ is an appropriate characteristic value of A. The solution to non-linear systems of difference equations such as (3) is not usually easy.

We assume that A has a dominant characteristic value λ. This means that A has distinct real characteristic values which are reciprocal, since A is unimodular. We may assume that λ is positive for otherwise we could deal with $-A$. Hence

$$\lambda+\lambda^{-1}=\alpha+\delta=k, \text{ say, where } k>2.$$

We discuss an example first. We normalize the matrix

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

to a unimodular

$$A=A_1=\begin{bmatrix} 2/\sqrt{3} & -1/\sqrt{3} \\ -1\sqrt{3} & 2/\sqrt{3} \end{bmatrix},$$

with characteristic roots $\sqrt{3}$, $1/\sqrt{3}$.

Application of the relations (3) gives

	$n=1$	$n=2$	$n=3$	$n=4$
$\alpha_n/\sqrt{3}=$	$2/3$	$14/15$	$122/123$	$1094/1095$
$\gamma_n/\sqrt{3}=$	$-1/3$	$1/5$	$-3/41$	$9/365$

which indicates that $\alpha_n \to \sqrt{3}$, $\gamma_n \to 0$. The general form of α_n, γ_n can be conjectured from the above table and established by induction. We find

$$\frac{\alpha_n}{\sqrt{3}}=\frac{3\times 9^{n-1}+1}{3(9^{n-1}+1)}, \quad \frac{(-1)^n\gamma_n}{\sqrt{3}}=\frac{2\times 3^{n-1}}{3(9^{n-1}+1)},$$

so that the convergence of $\{\gamma_n\}$ is ultimately geometric with common ratio $1/3=\lambda^{-2}$ while that of $\{\alpha_n\}$ is ultimately geometric with common ratio $1/9=\lambda^{-4}$.

In the general real 2×2 case we can prove that

$$\left\{\begin{aligned} \alpha_{n+1}&=\frac{p(\alpha^2+\gamma^2+\alpha q)-(\alpha+q)}{(\alpha+q)^2+\gamma^2} \\ \gamma_{n+1}&=(-1)^n\frac{\gamma(pq+1)}{(\alpha+q)^2+\gamma^2} \end{aligned}\right. \tag{4}$$

where $v_n=\lambda^n-\lambda^{-n}$, $p=p_n=v_{n+1}/v_n$, $q=q_n=-v_{n-1}/v_n$. Since $p_n\sim\lambda$, $q_n\sim-\lambda^{-1}$ this gives $\alpha_n\to\lambda$, $\gamma_n\to 0$.

In order to establish (4) we write $x_n=\alpha_n$, $y_n=(-1)^n\gamma_n$ in (3) to get

$$x_{n+1}=k-(x_n/(x_n^2+y_n^2)),\; y_{n+1}=y_n/(x_n^2+y_n^2), \tag{5}$$

which we combine as $z_{n+1}=k-\bar{z}_n/(z_n\bar{z}_n)$, i.e.,

$$z_{n+1}=k-z_n^{-1}, n=1,2,\dots, \tag{6}$$

where $z_r=x_r+iy_r$, $\bar{z}_r=x_r-iy_r$.

We have therefore reduced our problem to that of the iteration of the fractional linear transformation

$$w=\frac{kz-1}{z}. \tag{7}$$

This is a well-known problem. An essentially geometric solution is given, e.g., by T. J. I'A. BROMWICH [10, p. 22, ex. 4]. This depends on the fact that (7) can be represented in the form

$$\frac{w-\lambda}{w-\lambda^{-1}}=\lambda^{-2}\left(\frac{z-\lambda}{z-\lambda^{-1}}\right) \tag{7'}$$

which gives

$$\frac{z_{n+1}-\lambda}{z_{n+1}-\lambda^{-1}}=\lambda^{-2n}\left(\frac{z_1-\lambda}{z_1-\lambda^{-1}}\right),$$

so that $z_n\to\lambda$, as required. We can derive (7′) using the fact that a fractional linear transformation with fixed points at $0,\infty$ is necessarily linear, or by using the cross-ratio property of fractional linear transformations. For details compare, e.g. CARATHÉODORY [11, p. 14] or KREYSZIG [12, p. 503].

A second method is simply to establish, by induction,

$$z_{n+1}=\frac{pz_1+1}{z_1+q} \tag{8}$$

where $z_1=a-iy$ and $p=p_n$, $q=q_n$ are as defined above. Taking the real and imaginary parts of (8) gives (4).

Our third method, preferable in the matrix context, follows. We begin by recalling that if

$$W=\frac{aw+\beta}{\gamma w+\delta},\ w=\frac{az+b}{cz+d}$$

then

$$W=\frac{Az+B}{Cz+D}\quad\text{where}\quad \begin{bmatrix}A & B\\ C & D\end{bmatrix}=\begin{bmatrix}a & \beta\\ \gamma & \delta\end{bmatrix}\ \begin{bmatrix}a & b\\ c & d\end{bmatrix}.$$

Thus the iteration of a fractional linear transformation is equivalent to the powering of a matrix.

We use the following result.

Lemma. If M is a 2×2 matrix with distinct characteristic values λ, μ then

$$M^n=\begin{bmatrix} ad\lambda^n-bc\mu^n & -ab(\lambda^n-\mu^n) \\ cd(\lambda^n-\mu^n) & -bc\lambda^n+ad\mu^n \end{bmatrix} \tag{9}$$

where

$$T=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a unimodular matrix which diagonalizes M.

Proof. Either by induction or let $D=\operatorname{diag}\,[\lambda,\ \mu]$ so that $T^{-1}MT=D$, $M=TDT^{-1}$ and

$$\begin{aligned} M^n &= \{TDT^{-1}\}\{TDT^{-1}\}\ldots\{TDT^{-1}\} \\ &= TD^nT^{-1} \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned}$$

which gives (9), on multiplying out.

We now observe that when $A=\begin{bmatrix} k & -1 \\ 1 & 0 \end{bmatrix}$ then

$$T=(\lambda-\lambda^{-1})^{-1/2}\begin{bmatrix} \lambda & \lambda^{-1} \\ 1 & 1 \end{bmatrix},$$

where λ is as already defined. The rest of the discussion is a matter of elementary algebra.

Writing as before, $v_n=\lambda^n-\lambda^{-n}$, we find from (9)

$$z_{n+1}\equiv x_{n+1}+iy_{n+1}=\frac{v_{n+1}(x_1+iy_1)-v_n}{v_n(x_1+iy_1)-v_{n-1}}.$$

Multiplying above and below on the right by $v_n(x_1-iy_1)-v_{n-1}$ and equating real and imaginary parts shows that

$$\begin{cases} x_{n+1}=\{v_nv_{n+1}(x_1^2+y_1^2)+v_nv_{n-1}-x_1(v_n^2+v_{n+1}v_{n-1})\}/D, \\ y_{n+1}=(v_n^2-v_{n+1}v_{n-1})y_1/D, \end{cases}$$

where

$$D = v_n^2(x_1^2 + y_1^2) - 2v_{n-1}v_n x_1 + v_{n-1}^2 .$$

This is another form of (4). We find that, as $n \to \infty$,

$$\begin{cases} x_{n+1} - \lambda = [2(\lambda - \lambda^{-1})\{a^2 + \gamma^2 + 1 - ka\} + O(\lambda^{-2n})]/D, \\ y_{n+1} = (\lambda - \lambda^{-1})^2 \gamma / D, \end{cases} \tag{10}$$

where

$$D = [(a - \lambda^{-1})^2 + \gamma^2]\lambda^{2n} + O(1) = O(\lambda^{2n}).$$

The relations (10) establish the convergence of the *QR*-process.

Note that when the matrix A is symmetric, as well as unimodular, we have $a\delta - \gamma^2 = 1$, i.e., $a(k-a) - \gamma^2 = 1$, i.e. $a^2 + \gamma^2 + 1 = ka$ so that (10) gives $x_n - \lambda = O(\lambda^{-4n})$, $y_n = O(\lambda^{-2n})$, in agreement with the numerical results in the special case.

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