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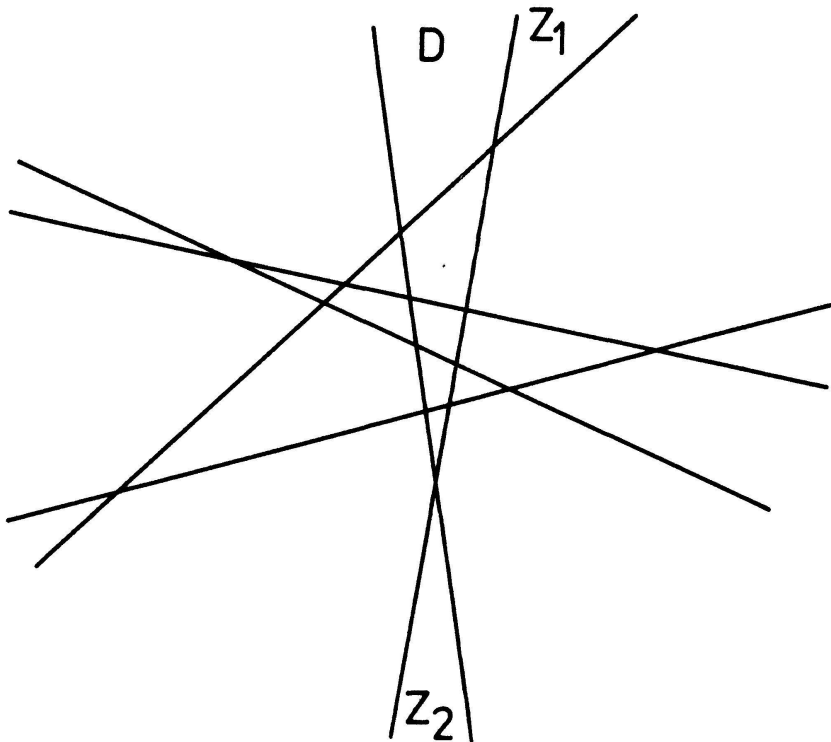
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Abb.2. Eine Anordnung $A_{V,E}(6)$.

struierten einfachen Anordnung $A_{V,E}(n)$ folgt etwa aus den verschiedenen Schnittpunktzahlen.

Damit ist Satz 3 bewiesen. Es kann ausserdem gezeigt werden, dass für $n \equiv 0 \pmod{4}$ weitere maximale Anordnungen $A_{V,E}(n)$ existieren, und auch für $n \not\equiv 0 \pmod{4}$ gibt es, bis auf einige Ausnahmen, nichtisomorphe Anordnungen $A_{V,E}(n)$.

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LITERATURVERZEICHNIS

- 1 B. Grünbaum: Arrangements und spreads. Am. Math. Soc., Providence, R.I. (1972).
- 2 H. Harborth und I. Mengersen: Geradenanordnungen mit maximaler Anzahl vierseitiger Flächen (eingereicht).

Kleine Mitteilungen

An application of Dirichlet convolution in proving some inequalities from elementary number theory

1. The aim of this note is to show how we can use the properties of a ring of real arithmetic functions to prove some known inequalities from the elementary theory of numbers.

We denote by f an arithmetic function and by $f(n)$ the value of f at n . Let A be the set of all real arithmetic functions. If $f, g \in A$, then their Dirichlet convolution is defined by

$$(f * g)(n) = \sum_{k|n} f(k)g(n/k), \quad n \geq 1.$$

It is well known that A forms a commutative ring with convolution as multiplication and with the usual addition. This ring has the zero and unit element Θ and e , respectively, defined by

$$\Theta(n) = 0, \quad n \geq 1; \quad e(n) = \begin{cases} 1 & \text{for } n = 1 \\ 0 & \text{for } n \neq 1. \end{cases}$$

We further define the product fg as usual,

$$(fg)(n) = f(n)g(n), \quad n \geq 1.$$

We write $f \leq g$ ($f \geq g$) when $f(n) \leq g(n)$ ($f(n) \geq g(n)$) for $n \geq 1$. We use in the proofs of all inequalities under consideration the following trivial lemma:

Lemma. *If $f, g \in A$ and $f \geq \Theta, g \geq \Theta$, then $f * g \geq \Theta$.*

2. We shall denote by $d(n)$ and $\sigma_k(n)$ the number of divisors and the sum of k -th powers of divisors of n respectively ($\sigma_1(n) = \sigma(n)$), by μ and φ_k the Möbius function and the Jordan function, respectively ($\varphi_k(n) = n^k \prod_{p|n} (1 - p^{-k})$, p prime, is defined for positive integers k, n as the number of different sequences a_1, \dots, a_k containing k (equal or distinct) positive integers $\leq n$ such that $(a_1, \dots, a_k, n) = 1$; $\varphi_1 = \varphi$ is the Euler function). For integral k , define the arithmetic function I_k by

$$I_k(n) = n^k, \quad n \geq 1$$

($I_0 = I$). In the proofs we shall use the following known equalities valid in the ring A :

$$\begin{aligned} d &= I * I, & (1) & & I * \mu &= e, & (2) & & I * \varphi_k &= I_k, & (3) \\ \varphi_k I_{-k} &= I * (\mu I_{-k}), & (4) & & \sigma_k I_{-k} &= I * I_{-k}, & (5) & & \sigma_k &= \varphi_k * d. & (6) \end{aligned}$$

The first inequality is due to Bagchi and Gupta [1]:

2.1. If $n > 1$ then $\sigma(n) \geq \varphi(n) + d(n)$.

Proof: It suffices to show that $\sigma + e \geq \varphi + d$, that is $\sigma + e - \varphi - d \geq \Theta$. From (6) for $k = 1$ we obtain

$$\sigma + e - \varphi - d = \varphi * d + e - \varphi - d = (\varphi - e) * (d - e).$$

Since $\varphi - e \geq \Theta$ and $d - e \geq \Theta$ the inequality $(\varphi - e) * (d - e) \geq \Theta$ holds by the lemma and the proof of (2.1) is completed.

Remarks: It is easy to notice that $((\varphi - e) * (d - e))(n) = 0$ holds iff $n = 1$ or n is a prime number. Hence $\sigma(n) > \varphi(n) + d(n)$ iff n is a composite number. We can get the inequality (2.1) from the relation $\sigma = \varphi * d$ also as follows: if $n > 1$ then

$$\sigma(n) = \sum_{k|n} \varphi(k) d(n/k) \geq \varphi(1) d(n) + \varphi(n) d(1) = \varphi(n) + d(n).$$

The second inequality is due to Makowski [2]:

2.2. For $n, k \geq 1$ we have $\varphi_k(n) + \sigma_k(n) \geq 2n^k$.

Proof: From the relations (6) and (3) we have

$$\varphi_k + \sigma_k - 2I_k = \varphi_k + (\varphi_k * d) - (\varphi_k * 2I) = \varphi_k * (e + d - 2I).$$

Applying the lemma we obtain inequality (2.2) immediately.

Remark: As (2.1) it is easy to verify that the strong inequality $\varphi_k(n) + \sigma_k(n) > 2n^k$ holds iff n is a composite number.

The third inequality is due to Makowski [2]:

2.3. If $k \geq 1, n > 1$ then $\varphi_k(n) + \sigma_k(n) \leq n^k d(n)$.

Proof: It suffices to prove the inequality

$$\varphi_k I_{-k} + \sigma_k I_{-k} \leq d + e.$$

Using the relations (1), (2), (4), (5) and the lemma we have

$$d + e - \sigma_k I_{-k} - \varphi_k I_{-k} = I * ((I + \mu) - (I + \mu) I_{-k}) = I * ((I + \mu)(I - I_{-k})) \geq \emptyset$$

and (2.3) follows.

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- 1 H. das Bagchi and M. Gupta: Problem 343. Jber. Dt. Math. Verein. 57, 8-9 (italics) (1954).
- 2 A. Makowski: Problem 339. El. Math. 15, 39-40 (1960).

Aufgaben

Aufgabe 804. Man bestimme die Anzahl der inkongruenten ebenen Netze eines regulären Ikosaeders. [Vgl. M. Jeger: Über die Anzahl der inkongruenten ebenen