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Diophantine representation of generalized Fibonacci numbers

Let A and B be integers such that either $A>0$ and $B=-1$ or $A>3$ and $B=1$. We define a sequence $R=\{R_n\}_{n=0}^{\infty}$ by the integers $R_0=0$, $R_1=1$ and the recurrence

$$R_n = AR_{n-1} - BR_{n-2}, \qquad n>1.$$

When $A=-B=1$ the positive terms of sequence R are the Fibonacci numbers; when $A=-B=1$, $R_0=2$, $R_1=1$ they are the Lucas numbers.
Jones [1, 2] has proved that the set of all Fibonacci (respectively Lucas) numbers is identical with the set of positive values of the polynomial

$$y(2-(y^2-yx-x^2)^2),$$

respectively

$$y(1-((y^2-yx-x^2)^2-25)^2),$$

as the variables x and y range over the positive integers.
The purpose of this paper is to extend the results of Jones on Fibonacci numbers to the generalized sequence. We prove two theorems and a lemma.

Theorem 1. *For non-negative integers x, y*

$$|x^2-Axy+By^2|=1 \tag{1}$$

if and only if x and y are consecutive terms of sequence R.

Theorem 2. *The set of all terms of sequence R is identical with the set of all non-negative values of the polynomial*

$$f(x,y)=y(2-(x^2-Axy+By^2)^2)$$

as the variables x and y range over the non-negative integers.

Lemma. *For every non-negative integer n,*
$$|R_{n+1}^2 - AR_{n+1}R_n + BR_n^2| = 1.$$

Proof of the lemma: The lemma is obviously true for $n=0$. And by definition of sequence R,

$$|R_{n+1}^2 - AR_{n+1}R_n + BR_n^2| = |(AR_n - BR_{n-1})^2 - A(AR_n - BR_{n-1})R_n + BR_n^2|$$
$$= |B(R_n^2 - AR_nR_{n-1} + BR_{n-1}^2)| = |R_n^2 - AR_nR_{n-1} + BR_{n-1}^2|$$

for $n>0$, since $|B|=1$.

Proof of theorem 1: Equality (1) holds for $x=R_{n+1}$, $y=R_n$ by the lemma. Thus we must prove that if (1) holds for integers $x,y \geqq 0$, then x and y are consecutive terms of sequence R.

Suppose that for integers $x_0, y_0 > 0$ we have

$$x_0^2 - Ax_0y_0 + By_0^2 = \varepsilon, \tag{2}$$

where $\varepsilon = 1$ or $\varepsilon = -1$. If $x_0, y_0 > 0$, we may assume $x_0 \geqq y_0$. Indeed, for $B=1$ condition (1) is symmetric in x and y; and for $B=-1$, (2) gives

$$x_0^2 - y_0^2 \geqq A\,x_0y_0 - 1 \geqq 0$$

(because $x_0y_0 \neq 0$).

Furthermore, if $x_0 \geqq y_0 > 0$, then

$$x_0 = \frac{1}{2}\left(Ay_0 + \sqrt{A^2y_0^2 - 4By_0^2 + 4\varepsilon}\right).$$

For if

$$x_0 = \frac{1}{2}\left(Ay_0 - \sqrt{A^2y_0^2 - 4By_0^2 + 4\varepsilon}\right)$$

and $y_0 > 0$, then in case $B=-1$, $A>0$ we would get $x_0 \leqq 0$; and in case $B=1$, $A>3$ the condition $x_0 \geqq y_0$ would imply the inequality

$$Ay_0 - 2y_0 \geqq \sqrt{A^2y_0^2 - 4y_0^2 + 4\varepsilon},$$

equivalent to the inequality

$$(2-A)y_0^2 \geqq \varepsilon,$$

which is impossible for $A>3$, $y_0>0$ and $\varepsilon = \pm 1$.

It follows from (2) that the integers $x_1=y_0$, $y_1=ABy_0-Bx_0$ also satisfy equation (1) since $B^2=1$ gives

$$x_1^2-Ax_1y_1+By_1^2=y_0^2-Ay_0(ABy_0-Bx_0)+B(ABy_0-Bx_0)^2$$
$$=Bx_0^2-ABx_0y_0+y_0^2=B(x_0^2-Ax_0y_0+By_0^2)=B\varepsilon.$$

But if $x_0>0$ and $y_0>0$, then $x_1>0$ and

$$y_1=ABy_0-Bx_0=ABy_0-\frac{B}{2}(Ay_0+\sqrt{A^2y_0^2-4By_0^2+4\varepsilon})$$

$$=\frac{B}{2}(Ay_0-\sqrt{A^2y_0^2-4By_0^2+4\varepsilon}\geqq 0 \tag{3}$$

($y_1=0$ only if $y_0=1$).
We show next that $y_1<y_0$, except perhaps when $A=-B=1$, $y_0=1$. In case $B=1$, $A>3$, using the form of y_1 given in (3), we get

$$y_1=\frac{1}{2}(Ay_0-\sqrt{A^2y_0^2-4y_0^2+4\varepsilon})<\frac{1}{2}(Ay_0-\sqrt{A^2y_0^2-4Ay_0^2+4y_0^2})=y_0$$
since $A^2y_0^2-4y_0^2+4\varepsilon>A^2y_0^2-4Ay_0^2+4y_0^2$, i.e. $(A-2)y_0^2>-\varepsilon$, clearly holds for $y_0>0$. And if $B=-1$, then

$$y_1=\frac{1}{2}(\sqrt{A^2y_0^2+4y_0^2+4\varepsilon}-Ay_0)<\frac{1}{2}((A+2)y_0-Ay_0)=y_0$$

since $4\varepsilon<4Ay_0^2$, except perhaps if $A=1, y_0=1$.
Continuing this procedure we construct the strictly decreasing sequences $y_0, y_1, y_2, \ldots$ and $x_0, x_1, x_2, \ldots$, where

$$x_i=y_{i-1} \quad \text{and} \quad y_i=ABy_{i-1}-Bx_{i-1} \quad \text{for} \quad i>0 \tag{4}$$

and $x_i>y_i\geqq 0$, if $y_{i-1}>0$ (except perhaps if $y_{i-1}=1$ in case $A=-B=1$). Furthermore equality (1) holds for $x=x_i$, $y=y_i$.
The construction comes to an end when an index j is reached such that $y_j=0$ (or $y_j=1$ in case $A=-B=1$). If $y_j=0$, then $x_j=1$, so that $y_j=R_0$ and $x_j=R_1$. But by (4) we can show that if $y_i=R_k$ and $x_i=R_{k+1}$ for some indices i and k, then $y_{i-1}=R_{k+1}$ and $x_{i-1}=Ay_{i-1}-By_i=AR_{k+1}-BR_k=R_{k+2}$ (since $B^2=1$); this shows that y_0, x_0 are also consecutive terms of sequence R. If $A=-B=1$ and $y_j=1$ for some index j, then $x_j=2$, 1 or 0. But $(y_j,x_j)=(1,2)=(R_2,R_3)$, $(y_j,x_j)=(1,1)=(R_1,R_2)$ and $y_j=1$, $x_j=0$ imply that $y_{j-1}=0=R_0$, $x_{j-1}=1=R_1$; therefore we get as above that y_0, x_0 are also consecutive terms of sequence R.
This completes the proof of theorem 1.

Proof of theorem 2: Because of the conditions imposed on A and B, we have

$$x^2 - Axy + By^2 = 0$$

for integers x and y if and only if $x = y = 0$. Therefore by theorem 1 for non-negative integers x, y, we have $f(x,y) = 0$ if and only if $y = 0$, $f(x,y) = y > 0$ if and only if x and $y > 0$ are two consecutive terms of the sequence R, and $f(x,y) < 0$ in any other cases.

Remark: One can easily see that theorem 1 is valid for cases $A = 1$, $B = 1$ and $A = 2$, $B = 1$, but sequence R is degenerate in these cases. In case $A = 3$, $B = 1$ theorem 1 is false since $x = 2$, $y = 1$ is a solution of equation (1) and 2 is not a term of sequence R.

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REFERENCES

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Kleine Mitteilungen

Eine merkwürdige Familie von beweglichen Stabwerken

1. Sei $ABA'B'$ ein *gelenkiges Antiparallelogramm* mit den Seitenlängen $AB = A'B' = a$ und $AB' = A'B = d > a$. Wird es in seiner Ebene so bewegt, dass der Schnittpunkt O der Langseiten und die Symmetrieachse z festbleiben (Fig. 1), dann rollt bekanntlich eine Ellipse e mit den Brennpunkten A, B und der Hauptachse d auf einer kongruenten Ellipse e' mit den Brennpunkten A', B' gleitungslos ab, wie die Betrachtung des gemeinsamen Linienelements (O, z) lehrt; diese Tatsache bildet die kinematische Grundlage für elliptische Zahnräder [2]. Alle vier Gelenke des Antiparallelogramms wandern dabei auf einer gemeinsamen, aus zwei kongruenten Ovalen bestehenden Bahnkurve 6. Ordnung, wie in [5] gezeigt wurde.
Bezeichnet $r = OA$ den Radiusvektor des Punktes A und ψ den Richtungswinkel, gemessen von der zur z-Achse normalen x-Achse aus, so hat A die kartesischen Koordinaten $x = r\cos\psi$, $z = r\sin\psi$ und B die Koordinaten $\bar{x} = (d-r)\cos\psi$, $\bar{z} = (r-d)\sin\psi$. Die auf $AB = a$ bezügliche Distanzformel liefert dann für die *Bahnsextik* k die Polargleichung

$$r(d-r)\cos^2\psi = m^2 \quad \text{mit} \quad 4m^2 = d^2 - a^2, \tag{1}$$

welche auf die kartesische Gleichung

$$(x^2+z^2)(x^2+m^2)^2 = d^2x^4 \tag{2}$$