Zeitschrift: Elemente der Mathematik

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 38 (1983)

Heft: 1

Artikel: Regular graphs containing a given graph

Autor: Akiyama, Jin / Era, Hiroshi / Harary, Frank

DOI: https://doi.org/10.5169/seals-37177

Der Grundgedanke dieses Beweises ist also, zu zeigen, dass diejenigen Paare (a,b), deren Schrittzahl $T(a,b)$ die Abschätzung von Dixon (Lemma 2) erfüllen, den Hauptbeitrag zu dem gesuchten Mittelwert liefern.
Zum Abschluss seien noch einige numerische Ergebnisse angegeben, wie man sie mit einem programmierbaren Taschenrechner errechnen kann. Die Zahlen der Tabelle sprechen wohl für sich.

N	T_N	$\frac{12}{\pi^2}\log 2\log N$
100	3.98	3.88
200	4.55	4.47
300	4.89	4.81
400	5.13	5.05
500	5.31	5.24
1000	5.90	5.82

Der Wert für $n=1000$ wurde durch Simulation (Monte-Carlo-Methode) gewonnen [7].

Hans Kilian, Dortmund

LITERATURVERZEICHNIS

1 R.P. Boas, J.R.: Partial sums of infinite series, and how they grow. Am. Math. Monthly *85*, 237–258 (1978).
2 J.D. Dixon: The number of steps in the Euclidean algorithm. J. Number Theory *2*, 414–422 (1970).
3 J.D. Dixon: A simple estimate for the number of steps in the Euclidean algorithm. Am. Math. Monthly *78*, 374–376 (1971).
4 H. Heilbronn: The average length of a class of finite continued fractions. In: Abhandlungen zur Zahlentheorie und Analysis. Hrsg. Paul Turán. Plenum Press, New York 1968.
5 D.E. Knuth: The Art of Computer Programming, Vol.2. Seminumerical Algorithms. Addison-Wesley Publishing Company, Reading, Mass., 1969.
6 J.W. Porter: On a theorem of Heilbronn. Mathematika *22*, 20–28 (1975).
7 H. Kilian: Stochastische Überlegungen in zahlentheoretischen Zusammenhängen – 4 Beispiele. In: Schriftenreihe Didaktik der Mathematik der Universität für Bildungswissenschaften Klagenfurt, Bd.3, 1980.

 0013-6018/83/010011-05$1.50 + 0.20/0

Regular graphs containing a given graph

In the first book on graph theory ever written, Dénes König [4] proved that for every graph G, of order p and maximum degree d, there is a d-regular graph H containing G as an induced subgraph. Paul Erdös and Paul Kelly [1] solved the extremal problem of determining for a given graph G, the exact minimum number of points which must be added to obtain such a supergraph H. Our purpose is to study the corresponding problem where G is a subgraph of H which is not necessarily induced. We prove that at most $d+2$ new points are needed, and that this bound which is independent of p is best possible.

Let G be a graph of order p and maximum degree $\Delta(G)=d$. We follow the notation and terminology of [3]. What is the least possible order of a graph H which is regular of order d and which contains G as a subgraph? One can regard the problem in the following way. The graph G is given together with a set I of m new isolated points (m standing for more). A graph H is formed from G and I by adding joins (new lines) between two points in I, two points of G or between points in I and G. It is desired to make H regular of degree d and to have m as small as possible. In figure 1 we illustrate such a completion for two graphs, G_1, G_2 whose lines are drawn solid. The new lines of H_1, H_2 are drawn dashed.

Denote the degree of v_i in G by d_i, and call $f_i = d - d_i$ the *deficiency* of v_i, that is, the number of joins needed to complete v_i to degree d. Finally, call the numbers $s=\sum f_i$ the *total deficiency*.

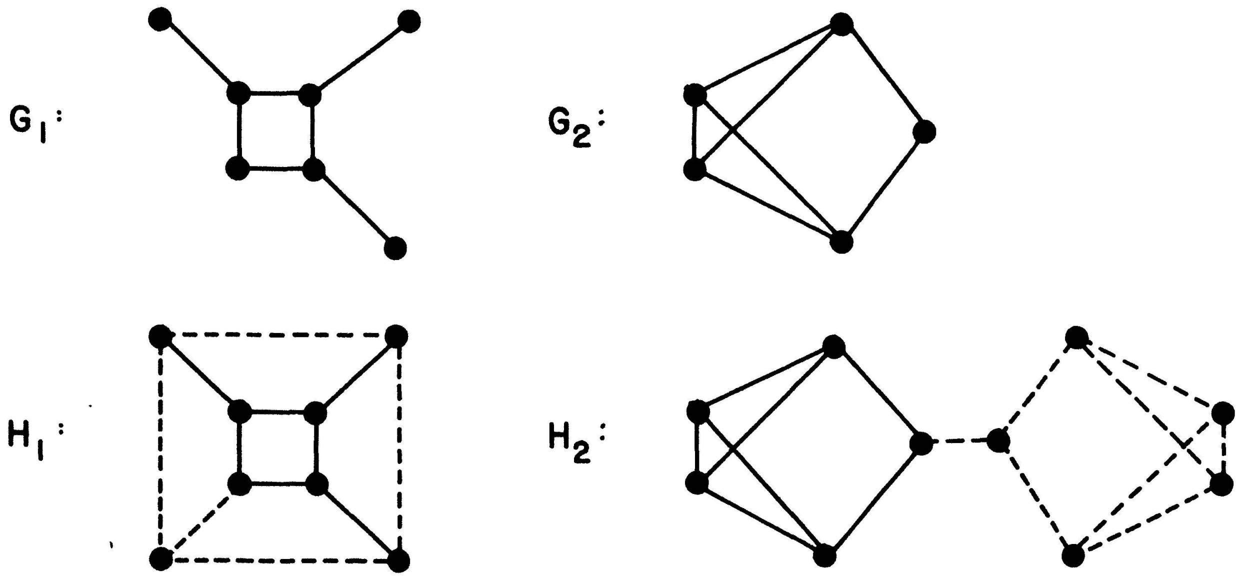

FIGURE 1.

Theorem. *If G is a graph of order p with maximum degree d, then it has a d-regular supergraph H with $d+2$ new points, and this is best possible.*

Proof: If there are two nonadjacent point v_i, v_j in G with $d_i, d_j < d$, join them by a new line. Continue this process until no such point pairs remain and write G' for the resulting spanning supergraph of G. Let $d'(u)$ be the degree of point u in G'. Thus $\Delta(G')=d$ and for every pair of nonadjacent points u, v, either $d'(u)=d$ or $d'(v)=d$. Hence the subgraph P induced in G' by all points u with $d'(u)<d$ must be complete, and its order t is at most $d-1$.

We prove the theorem only in the case that both p and d are odd, since the other cases can be readily proved similarly. In this case, it follows at once that the total deficiency s is odd.

First we note that as d is odd, K_{d+1} has a 1-factorization ([3], p. 85) into d 1-factors $F_1, F_2, \dots, F_d$. We now put $H' = G' \cup K_{d+1} \cup \{w\}$, so that H' has $d+2$ new points. Let $e_1, e_2, \dots, e_t$ be the deficiencies in H' of the points $u_1, u_2, \dots, u_t$ of P. Then the removal of $[e_i/2]$ lines of F_i from H' results in $2[e_i/2]$ points of degree $d-1$ in H'. Join u_i with these points of degree $d-1$ for each u_i in P. Join u_i with w if e_i is odd.

As the result of this procedure, all the points of H' other than w have degree d and the degree r of the point w is odd, since the total deficiency s is odd. Finally we remove any $(d-r)/2$ lines of the 1-factor F_{t+1} from H' and join w with the resulting $d-r$ points of degree $d-1$. The graph so constructed is d-regular, contains G, and has order $p+d+2$.

We now show by a family of examples that $d+2$ is best possible. Let G be obtained from the complete graph K_{p-1} with $p \geq 5$ by subdividing just one line by the insertion of a new point of degree 2; the graph G_2 in figure 1 illustrates $p=5$. Then we can readily see that if p is odd, at least $d+2$ new points are needed to construct a d-regular supergraph of G. □

Remark 1. The smallest d-regular supergraph H will of course depend on the structure of G and its order can range between p and $p+d+2$.

Remark 2. When pr is even, the minimum order of an r-regular supergraph H will range between p and $p+d+1$. Thus the bound in the theorem is decreased by 1 in this case.

The strengthening of the theorem in the following statement is easily accomplished by a proof which we omit as it is entirely analogous.

Corollary. *Let G be a graph of order p with maximum degree d, and r be an integer such that* $d \leq r \leq p-2$. *Then G has an r-regular supergraph of order at most* $p+r+1$ *or* $p+r+2$ *if* pr *is even or odd, respectively.* □

Jin Akiyama, Hiroshi Era and Frank Harary,
Nippon Ika University, Kawasaki, Japan; University of Michigan, Ann Arbor, USA

REFERENCES

1 P. Erdös and P. Kelly: The minimal regular graph containing a given graph. Am. Math. Monthly *70*, 1074–1075 (1963).
2 P. Erdös and P. Kelly: The minimal regular graph containing a given graph. In: A Seminar on Graph Theory, p. 65–69. F. Harary, ed. Holt, New York 1967.
3 F. Harary: Graph Theory. Addison-Wesley, Reading 1969.
4 D. König: Theorie der endlichen und unendlichen Graphen. Leipzig 1936, reprinted, Chelsea, New York 1950.

 0013-6018/83/010015-03$1.50 + 0.20/0

A remark on a paper by A. Grytczuk

In [3] Grytczuk showed that if $c_k(n)$ denotes the Ramanujan trigonometric sum, then

$$\sum_{d|k} |c_d(n)| = 2^{\omega(k/(k,n))}(k,n), \tag{1}$$

where $\omega(m)$ denotes the number of distinct prime divisors of m. In this note we prove a generalization of (1).