Zeitschrift: Elemente der Mathematik

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 38 (1983)

Heft: 1

Artikel: A remark on a paper by A. Grytczuk

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DOI: https://doi.org/10.5169/seals-37178

As the result of this procedure, all the points of H' other than w have degree d and the degree r of the point w is odd, since the total deficiency s is odd. Finally we remove any $(d-r)/2$ lines of the 1-factor F_{t+1} from H' and join w with the resulting $d-r$ points of degree $d-1$. The graph so constructed is d-regular, contains G, and has order $p+d+2$.
We now show by a family of examples that $d+2$ is best possible. Let G be obtained from the complete graph K_{p-1} with $p\geq 5$ by subdividing just one line by the insertion of a new point of degree 2; the graph G_2 in figure 1 illustrates $p=5$. Then we can readily see that if p is odd, at least $d+2$ new points are needed to construct a d-regular supergraph of G. □

Remark 1. The smallest d-regular supergraph H will of course depend on the structure of G and its order can range between p and $p+d+2$.
Remark 2. When pr is even, the minimum order of an r-regular supergraph H will range between p and $p+d+1$. Thus the bound in the theorem is decreased by 1 in this case.
The strengthening of the theorem in the following statement is easily accomplished by a proof which we omit as it is entirely analogous.

Corollary. *Let G be a graph of order p with maximum degree d, and r be an integer such that $d\leq r\leq p-2$. Then G has an r-regular supergraph of order at most $p+r+1$ or $p+r+2$ if pr is even or odd, respectively.* □

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 0013-6018/83/010015-03$1.50+0.20/0

A remark on a paper by A. Grytczuk

In [3] Grytczuk showed that if $c_k(n)$ denotes the Ramanujan trigonometric sum, then

$$\sum_{d\mid k} |c_d(n)| = 2^{\omega(k/(k,n))}(k,n), \tag{1}$$

where $\omega(m)$ denotes the number of distinct prime divisors of m. In this note we prove a generalization of (1).

Let h be an arithmetic function and if m is a natural number, let

$$h_m(n)=\begin{cases} h(n) & \text{if } n\mid m \\ 0 & \text{else}\end{cases}$$

Define

$$H_m(n)=\sum_{d\mid (m,n)} \mu(m/d)\,h(d)=\sum_{d\mid m} \mu(m/d)\,h_n(d). \tag{2}$$

Note that $H_1(n)=h(1)$ and if $a\geq 1$, then

$$H_{p^a}(n)=h_n(p^a)-h_n(p^{a-1})$$
$$=\begin{cases} h(p^a)-h(p^{a-1}) & \text{if } p^a\mid n \\ -h(p^{a-1}) & \text{if } p^{a-1}\,\|\, n. \\ 0 & \text{if } p^{a-1}\nmid n\end{cases}$$

We now show that $H_m(n)$ is a multiplicative function of m if $h(m)$ is also. To do this it will suffice to show that $h_n(d)$ is a multiplicative function of d, since, by (2), $H_m(n)$ is then a convolution of two multiplicative functions. If $(m,n)=1$ and k is a natural number, then $mn\mid k$ if and only if $m\mid k$ and $n\mid k$. Thus, by the definition of $h_k(mn)$, we have

$$h_k(mn)=\begin{cases} h(mn) & \text{if } mn\mid k \\ 0 & \text{else}\end{cases}=\begin{cases} h(m)h(n) & \text{if } m\mid k \text{ and } n\mid k \\ 0 & \text{else}\end{cases}$$
$$=\big\{ h_k(m)\,h_k(n).$$

Thus $h_k(d)$ is a multiplicative function of d if h is multiplicative.

Theorem. *If h is a multiplicative function such that*

$$h(p^a)-h(p^{a-1})\geq 0 \tag{3}$$

for all $a\geq 1$ and primes p, then for all positive integers k and m, we have

$$\sum_{d\mid k} |H_d(n)|=2^{\omega(k/(k,n))}\,h\big((k,n)\big). \tag{4}$$

Proof: Since $H_d(n)$ is multiplicative so is $|H_d(n)|$ and thus so is the left hand side of (4) ([4], theorem 265). Since, for n fixed and $(k,l)=1$, we have $(kl,n)=(k,n)(l,n)$ (see [5], p. 17) and since h is multiplicative we see that the right hand side of (4) is also a multiplicative function of k. To prove the theorem it therefore suffices to show that (4) holds when k is a prime power, $k=p^a$.

Note that, by (3), $h(p^a)\geq h(p^{a-1})\geq\cdots\geq h(1)\geq 0$, since h multiplicative implies $h(1)=1$ or $h(1)=0$.

Suppose $p^b \| n$. If $0 \le b < a$, then

$$\sum_{j=0}^{a} |H_{p^j}(n)| = h(1) + \sum_{j=1}^{b} (h(p^j) - h(p^{j-1})) + |H_{p^{b+1}}(n)|$$
$$= h(1) + h(p^b) - h(1) + h(p^b) = 2h(p^b). \tag{5}$$

If $b \ge a$, then

$$\sum_{j=0}^{a} |H_{p^j}(n)| = h(1) + \sum_{j=1}^{a} (h(p^j) - h(p^{j-1})) = h(p^a). \tag{6}$$

If we compare (5) and (6) with the right hand side of (4), in each of the two cases we see that they agree. This proves our theorem.
Examples:
1. Let r be a positive integer and

$$C_k^{(r)}(n) = \sum_{\substack{x_1,\ldots,x_r \bmod k \\ (x_1,\ldots,x_r,k)=1}} \exp(2\pi i n(x_1 + \cdots + x_r)/k).$$

In [2], theorem 1, Cohen proves that

$$C_k^{(r)}(n) = \sum_{d|(k,n)} d^r \mu(k/d).$$

Thus, if we take $h(n) = n^r$, we have, by the theorem,

$$\sum_{d|k} |C_d^{(r)}(n)| = 2^{\omega(k/(k,n))} (k,n)^r.$$

The case $r = 1$ is (1), the result of Grytczuk.
2. Let $h(n) = d(n)$, the divisor function. Then d satisfies the hypotheses of the theorem and in this case

$$H_{p^a}(n) = \begin{cases} 1 & \text{if } p^a | n \\ -a & \text{if } p^{a-1} \| n. \\ 0 & \text{if } p^{a-1} \nmid n \end{cases}$$

The theorem states that in this case

$$\sum_{d|k} |H_d(n)| = 2^{\omega(k/(k,n))} d((k,n)).$$

It might be interesting to note that the evaluation of the sum $\sum_{d|k} H_d(n)$ is much easier even if h is not known to be multiplicative. Indeed it follows immediately from the Möbius inversion formula ([4], theorem 266) that

$$H_m(n) = \sum_{d|(m,n)} \mu(m/d) h(d) = \sum_{d|m} \mu(m/d) h_n(d)$$

if and only if

$$\sum_{d\mid k} H_d(n) = h_n(k) = \begin{cases} h(k) & \text{if} \quad k\mid n \\ 0 & \text{else} \end{cases}.$$

Finally we remark that it may be possible to generalize our result further to the class of functions considered by Anderson and Apostol in [1]. We hope to return to this in a later paper.

ACKNOWLEDGMENT

I would like to thank John Steinig for his suggestions for the improvement of the exposition of this paper.

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Nachtrag

P. Läuchli: Das Gitterspiel, El. Math. *37,* 109-113 (1982).

Nach Drucklegung des Aufsatzes stellte ich aufgrund eines neuen Buches [2], dessen Manuskript mir freundlicherweise von den Autoren zur Verfügung gestellt wurde, fest, dass das «Gitterspiel» von C. Berge schon 1907 von Wythoff beschrieben wurde [3]. Ferner hat Coxeter [1] eine sehr elegante Verbindung zum goldenen Schnitt gezogen.

P. Läuchli

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