Zeitschrift: Elemente der Mathematik

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 38 (1983)

Heft: 3

Artikel: The lattice polytope problem

Autor: Patruno, Gregg N.

DOI: https://doi.org/10.5169/seals-37187

LITERATURVERZEICHNIS

1 K. Fladt: Analytische Geometrie spezieller ebener Kurven. Frankfurt 1962.
2 G. Loria: Spezielle algebraische und transzendente ebene Kurven. Leipzig 1910.
3 H. Schmidt: Ausgewählte höhere Kurven. Essen 1949.
4 K. Strubecker: Differentialgeometrie I, Kurventheorie der Ebene und des Raumes. Berlin 1964.
5 K. Strubecker: Äquiforme Geometrie der isotropen Ebene. Arch. Math. *3*, 145-153 (1952).
6 K. Strubecker: Über die Parabeln zweiter bis vierter Ordnung I/II/III. PM *4*, 141-144, 169-174, 197-201 (1962).
7 K. Strubecker: Geometrie in einer isotropen Ebene. MNU *15*, 297-306, 343-351, 385-394 (1962/63), und Einführung in die höhere Mathematik, Bd. 1.
8 W. Vetter: Die Sichel des Archimedes. Eine Verallgemeinerung für die logarithmische Spirale. PM *24*, 54-56 (1982).
9 W. Vetter: Über die Peripheriewinkel der logarithmischen Spirale. Erscheint in PM.
10 H. Wieleitner: Spezielle ebene Kurven. Leipzig 1908.
11 W. Wunderlich: Darstellende Geometrie II. Mannheim 1967.
12 I. M. Yaglom: A simple non-euclidean geometry and its physical basis. New York 1979.
13 H. Zeitler: Über Brennkurven. DdM 1980, 1-11.

 0013-6018/83/030061-09$1.50+0.20/0

The lattice polytope problem

It has long been known ([6], p. 50) that the only regular polygons that can be embedded in the cubic lattice of E^n are the square (for $n \geqslant 2$), the triangle and the hexagon (both for $n \geqslant 3$). However, the analogous results for polytopes of higher dimension have not yet been fully described ([3], p. 46). In the present paper, we shall determine exactly which regular polytopes can be embedded in which regular polytopal lattices.

We shall be using the standard Schläfli notation, specifically:

(i) The symbol $\{n\}$ denotes a regular n-gon;

(ii) the regular n-dimensional polytope represented by $\{a_1, a_2, \ldots, a_{n-1}\}$ is a convex configuration of congruent $\{a_1, a_2, \ldots, a_{n-2}\}$'s, to be called *cells*, which fit together in such a way that each $(n-2)$-dimensional *face* belongs to two cells, and each $(n-3)$-dimensional *edge* to a_{n-1} cells.

(Thus the cube, having three squares meeting at each vertex, will be denoted by $\{4,3\}$.)

This notation is extended in the natural way to include the regular lattices; $\{6,3\}$, for example, refers to the hexagonal tiling of the plane.

The complete set of regular polytopes is given in Coxeter ([2], p. 292-295):

in E^2: $\{n\}$, with $n \geqslant 3$;
in E^3: $\{3,3\}, \{3,4\}, \{4,3\}, \{3,5\}, \{5,3\}$;
in E^4: $\{3,3,3\}, \{3,3,4\}, \{3,4,3\}, \{4,3,3\}, \{3,3,5\}, \{5,3,3\}$;
in E^n: with $n \geqslant 5$, $\{3_{n-1}\}, \{3_{n-2}, 4\}, \{4, 3_{n-2}\}$.

The regular lattices are ([2], p. 296):

in E^2: $\{4,4\}$, $\{3,6\}$, $\{6,3\}$;
in E^3: $\{4,3,4\}$;
in E^4: $\{4,3,3,4\}$, $\{3,4,3,3\}$, $\{3,3,4,3\}$;
in E^n, with $n \geqslant 5$: $\{4,3_{n-2},4\}$.

Lattice polytopes are already known for the following lattices:

$\{4,4\}$ - The only regular polygon constructible on a square lattice is the square itself [4].

$\{4,3_{n-2},4\}$, $n \geqslant 3$ - As was mentioned earlier, no polygon besides the triangle, square and hexagon can be embedded in a cubic lattice of any dimension. This implies similar negative results for the polytopes containing regular pentagons - $\{3,5\}$, $\{5,3\}$, $\{3,3,5\}$, $\{5,3,3\}$. Furthermore, a regular n-dimensional simplex can be inscribed in an n-dimensional cubic lattice if and only if $n+1$ is an odd square, the sum of two odd squares, or a multiple of four [6]. The 24 permutations of $(\pm 1, \pm 1, 0, 0)$ in Cartesian coordinates serve as the vertices of a $\{3,4,3\}$ ([5], p. 51).

The remaining polytopes are embedded routinely:

$\{6\}$: $(-1,0,1,0_{n-3})$, with the first three coordinates permuted;
$\{3_{k-1}\}$, with $2 \leqslant k \leqslant n-1$: any k permutations of $(1,0_{n-1})$;
$\{3_{k-2},4\}$, with $3 \leqslant k \leqslant n$: any k permutations of $(1,0_{n-1})$ and their negatives;
$\{4,3_{k-2}\}$, with $2 \leqslant k \leqslant n$: $(\pm 1_k, 0_{n-k})$.

$\{3,6\}$ - Here, the only lattice polygons are the triangle and the hexagon [1].

Of the three remaining lattices, $\{6,3\}$ is the most easily dealt with. Since the hexagonal lattice is contained in the triangular lattice (fig. 1), no polygons can be inscribed in $\{6,3\}$ that are not inscribable in $\{3,6\}$. (The triangle and the hexagon are readily found.)

For the last two lattices, we shall need the following preliminary results:

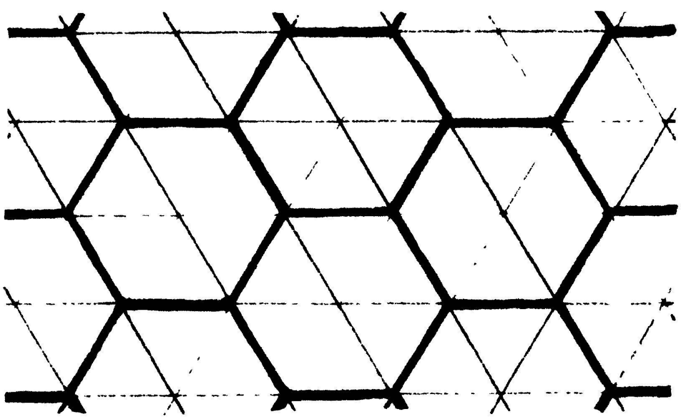

Figure 1

Lemma 1. *The points in E^4 with two odd and two even coordinates form a* $\{3,4,3,3\}$ *lattice* ([2], p. 158).

Lemma 2. *The points in E^4 whose coordinates are integers of the same parity form a* $\{3,3,4,3\}$ *lattice.*

Proof of lemma 2: Since the unit cells of the $\{3,4,3,3\}$ lattice have the same orientation in space ([2], p. 156), we can reach any of them by a translation of the cell centered at the origin - the one with vertices $(\pm 1, \pm 1, 0, 0)$ and permutations. The images of these 24 points under translation through (a,b,c,d) will have the required two odd and two even coordinates if and only if a,b,c,d are integers of the same parity. The points (a,b,c,d), being the centers of the translated cells, will form a reciprocal $\{3,3,4,3\}$ lattice.
It follows from lemmas 1 and 2 that if a polytope can be embedded in $\{3,4,3,3\}$ or in $\{3,3,4,3\}$, then it can also be embedded in the cubic lattice $\{4,3,3,4\}$. Conversely, by doubling the Cartesian coordinates of a polytope embedded in $\{4,3,3,4\}$ and translating the resulting figure either through $(1,1,0,0)$ or through $(0,0,0,0)$, we will obtain a similar polytope which satisfies the respective parity requirements of $\{3,4,3,3\}$ or $\{3,3,4,3\}$. Therefore:

Theorem. *The same polytopes can be embedded in each of the regular four-dimensional lattices.*

With this result, the lattice polytope problem is completely solved.

Gregg N. Patruno, 373 Giffords Lane, Staten Island, NY, USA

REFERENCES

1 O. Buggisch: Aufgabe 709. El. Math. *30,* 15 (1975).
2 H.S.M. Coxeter: Regular Polytopes, 2nd ed. Macmillan, New York 1963.
3 J. Hammer: Unsolved Problems Concerning Lattice Points. Research Notes in Mathematics, No. 15. Pitman, London, San Francisco, Melbourne 1978.
4 W. Scherrer: Die Einlagerung eines regulären Vielecks in ein Gitter. El. Math. *1,* 97-98 (1946).
5 L. Schläfli: Theorie der vielfachen Kontinuität. Denkschriften der Schweizerischen Naturforschenden Gesellschaft *38,* 1-237 (1901).
6 I.J. Schoenberg: Regular simplices and quadratic forms. J. Lond. Math. Soc. *12,* 48-55 (1937).

 0013-6018/83/030069-03$1.50+0.20/0

Aufgaben

Aufgabe 879. Welche Beziehung besteht zwischen dem Abstand und dem Winkel irgend zweier windschiefer Erzeugenden eines einschaligen Drehhyperboloids?

W. Wunderlich, Wien, A