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On L^2 and L^3

The purpose of this note is to show a few results of the elementary Lorentzian geometry. Our intention is to familiarize the reader with these topics since it is not usual to find papers on low-dimensional Lorentzian geometry.
In part I, after showing null points are not vertices of a curve, we have a proposition stating the analogue of the four vertex theorem but for null points.
In part II, in the tridimensional Lorentzian space we characterize null curves and the osculating sphere at each of its points.
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Preliminaries

Let L^n be the vector space R^n with the Lorentzian inner product

$$\langle x, y\rangle = -x_1 y_1 + \ldots + x_n y_n$$

for

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n).$$

The norm of x is $\| x \| = \sqrt{|\langle x, x\rangle|}$.
In L^n, a vector x is called null if $\langle x, x\rangle = 0$ and a null curve is a curve whose tangent vector at every point is null.
The reader will find information about the trigonometry of L^2 in [1].

On L^2

Let $\alpha: I \to L^2$ be a curve in the plane L^2, not null, and $\beta: J \to L^2$ be a reparametrization of $\alpha(I)$ such that $\| \beta'(s) \| = 1$. Calling $\frac{d\alpha}{dt} = \alpha'$, it is easy to check

$$\frac{dt}{ds} = \frac{1}{\| \alpha' \|}, \quad \frac{d^2t}{ds^2} = -\frac{\langle \alpha', \alpha''\rangle}{\| \alpha' \|^3}, \quad \beta'(s) = \alpha'(t) \cdot \frac{dt}{ds}$$

$$\beta''(s) = \frac{\alpha''\langle \alpha', \alpha'\rangle - \alpha'\langle \alpha', \alpha''\rangle}{\| \alpha' \|^{3/2}} = \frac{(\alpha' \wedge \alpha'') \wedge \alpha'}{\| \alpha' \|^{3/2}}.$$

Thus, the curvature $k(t)$ of $\alpha(t)$ is given by

$$\beta''(s) = \frac{\| \alpha' \wedge \alpha'' \|}{\| \alpha' \|^3} = k.$$

In terms of the coordinates of $\alpha(t) = (x(t), y(t))$ and keeping in mind the norm of L^2, we get

$$k(t) = \frac{x' y'' - x'' y'}{((x')^2 - (y')^2)^{3/2}}.$$

Consequence: Null points are not vertices of a curve.
In spite of the preceding paragraphs we obtain the analogue of the four vertex theorem.

Proposition: A simple, closed, convex curve has at least four null points.

Proof: Let $C = \alpha(t) = (x(t), y(t))$ be a simple, closed, convex curve in L^2 and let L be the curve given by the set of points $(x(t), x(t) + r)$ where r is a constant real number such that the set $L \cap C$ consist of two points. That is possible since C is a simple, convex curve.
Now, let M and N be those intersection points. Clearly, M and N are null points.
We assume the parametrization $C = \alpha(t)$ for $t \in [a, b]$ with $(x(a), y(a)) = M (x(c), y(c)) = N, c \in (a, b)$.
We know $x(a) = x(b)$, $y(a) = y(b)$, $x'(a) = y'(a)$ and $x'(c) = y'(c)$.
The segment $\overline{MN}$ does not intersect C.
We can consider the curve $\beta(t)$ given by

$$\beta(t) = \alpha(t) \cdot \begin{pmatrix} \cos\frac{-\pi}{4} & -\sin\frac{-\pi}{4} & x(c) \\ \sin\frac{-\pi}{4} & \cos\frac{-\pi}{4} & y(c) \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} x(t) - \frac{\sqrt{2}}{2} y(t) \\ \frac{\sqrt{2}}{2} x(t) + \frac{\sqrt{2}}{2} y(t) \\ 1 \end{pmatrix}$$

for $t \in [a, c]$. Of course we omit the last coordinate.

We check

$$\beta(a) = \left(\frac{\sqrt{2}}{2} x(a) - \frac{\sqrt{2}}{2} y(a), \frac{\sqrt{2}}{2} x(a) + \frac{\sqrt{2}}{2} y(a)\right) = (0, \sqrt{2}\, y(a))$$

$$\beta(c) = \left(\frac{\sqrt{2}}{2} x(c) - \frac{\sqrt{2}}{2} y(c), \frac{\sqrt{2}}{2} x(c) + \frac{\sqrt{2}}{2} y(c)\right) = (0, \sqrt{2}\, x(c))$$

$$\beta'(a) = (0, \sqrt{2}\, x'(a))$$

$$\beta'(c) = (0, \sqrt{2}\, x'(c))$$

then, we can apply Rolle's theorem and we have that there exists $t_0 \in (a, c)$ such that

$$\frac{\sqrt{2}}{2} x'(t_0) + \frac{\sqrt{2}}{2} y'(t_0) = 0.$$

Now, coming back to $\alpha(t)$ at t_0, we find $\alpha'(t_0)$ is a null vector, equivalently, $\alpha(t_0)$ is a null point.

Analogously, let $\gamma(t)$ be the rotation of $\alpha(t)$ for $t \in [c, d]$. We will obtain the null point which is the fourth one.

Remark: This result holds for the Lorentzian-Poincaré upper half plane, that is the upper half plane with $ds^2 = \dfrac{dx^2 - dy^2}{y^2}$.

On L^3

Let $\alpha = \alpha(s)$ be a null curve in L^3, where s is not the parameter propertime, [3].
A null frame in L^3 is an ordered triple of vectors $F = (E^1, E^2, E^3)$ such that

$$\langle E^1, E^1\rangle = \langle E^2, E^2\rangle = 0, \quad \langle E^1, E^2\rangle = -1, \quad \langle E^3, E^3\rangle = 1, \quad \langle E^1, E^3\rangle = 0,$$
$$\langle E^2, E^3\rangle = 0$$

and

$$\det\begin{pmatrix} E^1 \\ E^2 \\ E^3 \end{pmatrix} = \pm 1.$$

From [2], we get a few definitions which we need to relate the curve and the null frame $F = (E^1, E^2, E^3)$.
A null frame F is proper if

$$L(F) = \left(\frac{1}{\sqrt{2}}(E^1 + E^2), \frac{1}{\sqrt{2}}(E^1 - E^2), E^3\right) \in SO^+(1, 2).$$

Taking the matrix

$$N = \frac{1}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

a curve $s \to F(s)$ in $SO^+(1, 2)$. N is a proper frame curve, and a frame for a null curve $\alpha(s)$ is a proper frame curve $F(s) = (E^1(s), E^2(s), E^3(s))$ such that $\alpha'(s)$ is a positive scalar multiple of $E^1(s)$. The curve $\alpha(s)$ is said to be framed by $F(s)$.
Now, we will assume $\alpha(s)$ is framed by $F(s) = (E^1(s), E^2(s), E^3(s))$ with $E^1(s) = \alpha'(s)$ at any point of $\alpha(s)$.
Differentiating

$$\langle dE^i, E^i\rangle = 0 \qquad i = 1, 2, 3,$$

$$\langle dE^1, E^2\rangle = -\langle E^1, dE^2\rangle, \quad \langle dE^2, E^3\rangle = -\langle E^2, dE^3\rangle, \quad \langle dE^1, E^3\rangle = -\langle E^1, dE^3\rangle.$$

Thus, the structure equations are

$$dE^i = \sum_J w_j^i E^j$$

and verify

$$w_1^1 = w_2^2, \quad w_3^3 = w_1^2 = w_2^1 = 0, \quad w_3^1 = w_2^3, \quad w_1^3 = w_3^2.$$

On the other side, from [2] we know that the Frenet equations hold

$$dE^1 = \quad k_1 E^1 + k_2 E^3$$

$$dE^2 = -k_1 E^2 + k_3 E^3$$

$$dE^3 = \quad k_3 E^1 + k_2 E^2$$

where the curvature k_i, $i = 1, 2, 3$ are functions of s. Comparing both systems we have

$$k_1 = 0, \quad k_2 = w_3^1 = w_2^3, \quad k_3 = w_1^3 = w_3^2.$$

In order to know the curvature k_2 and k_3 we will compute the osculating sphere of $\alpha(s)$.
We call osculating sphere of a curve at a point p to the sphere which has contact of order $\geqq 3$ with the curve at p. Then, we look for a sphere of center c and radius r which has contact of order 3 at the point $p = \alpha(s_0)$. To simplify the notation we will call $p = X(s) = X$.
Now, the surface must hold

$$f(s) = \langle X - c, X - c \rangle - r^2, \quad f(s_0) = f'(s_0) = f''(s_0) = f'''(s_0) = 0.$$

Derivating and applying the Frenet equations we have

$$\langle X - c, E^1 \rangle = 0, \quad k_2 \langle X - c, E^3 \rangle = 0,$$

$$\langle X - c, k_2' E^3 + k_2 k_3 E^1 + (k_2)^2 E^2 \rangle = 0.$$

We get $k_2 = 0$ and k_3 arbitrary. Also $r = 0$ and $c = X(s_0)$.
Coming back to the Frenet equations, we obtain $dE^1 = 0$ and E^1 constant, which say that $\alpha(s)$ is linear.

Summarizing:

The null curves in L^3 are the null straight lines and its osculating sphere at every point is a cone centered on that point.

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REFERENCES

1 G. Birman and K. Nomizu: Trigonometry in Lorentzian geometry. Amer. Math. Monthly, vol. 91, No. 9, Nov. 1984.
2 L. K. Graves: Codimension one isometric immersions between Lorentz spaces. Transaction of the A.M.S., vol. 252, 1979.
3 R. K. Sach and H. Wu: General relativity for mathematicians. Springer, New York, 1977.

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Über Lehmers Mittelwertfamilie

Gegenstand dieser Note ist die für positive x und y sowie für reelle Parameter r definierte Mittelwertfamilie

$$L_r(x, y) = \frac{x^{r+1} + y^{r+1}}{x^r + y^r},$$

die die drei klassischen Mittelwerte:

das arithmetische Mittel: $L_0(x, y) = \frac{x+y}{2}$,

das geometrische Mittel: $L_{-1/2}(x, y) = \sqrt{xy}$ und

das harmonische Mittel: $L_{-1}(x, y) = \frac{2xy}{x+y}$

enthält. Von H. W. Gould und M. E. Mays [4] ist für L_r die Bezeichnung Lehmer Mittel gewählt worden. Zahlreiche interessante Eigenschaften von L_r findet man in [1–4], [8]. Die einparametrige Funktionenschar $L_r(x, y)$ ist ein Spezialfall der im Jahre 1938 von C. Gini [3] für positive $x_1, \ldots, x_n$ sowie für reelle Parameter r und s eingeführten Mittelwertfamilie

$$G(r, s; x_1, \ldots, x_n) = \left[\sum_{i=1}^{n} x_i^r \Big/ \sum_{i=1}^{n} x_i^s\right]^{1/(r-s)} \quad \text{für} \quad r \neq s,$$

$$G(r, r; x_1, \ldots, x_n) = \exp\left[\sum_{i=1}^{n} x_i^r \log(x_i) \Big/ \sum_{i=1}^{n} x_i^r\right].$$

Bemerkenswert ist eine vor kurzem von D. Farnsworth und R. Orr [2] veröffentlichte Note über Gini Mittel, in der unter anderem gezeigt wird, wie sich das Lehmer Mittel