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Zahlreiche Eigenschaften von E findet man in [5–7], [10] und in der dort zitierten Literatur.
Gould und Mays [4] haben gezeigt, dass die einzigen Mittelwerte, die sowohl $E(r, s; x, y)$ als auch $L_r(x, y)$ angehören, das arithmetische, das geometrische und das harmonische Mittel von x und y sind.

Der Redaktion möchte ich für Verbesserungsvorschläge herzlich danken.

Horst Alzer, Waldbröl

LITERATUR

1 E. F. Beckenbach: A class of mean value functions. Amer. Math. Monthly *57*, 1–6 (1950).
2 D. Farnsworth and R. Orr: Gini means. Amer. Math. Monthly *93*, 603–607 (1986).
3 C. Gini: Di una formula comprensiva delle medie. Metron *13*, 3–22 (1938).
4 H. W. Gould and M. E. Mays: Series expansions of means. J. Math. Anal. Appl. *101*, 611–621 (1984).
5 E. B. Leach and M. C. Sholander: Extended mean values. Amer. Math. Monthly *85*, 84–90 (1978).
6 E. B. Leach and M. C. Sholander: Extended mean values II. J. Math. Anal. Appl. *92*, 207–223 (1983).
7 E. B. Leach and M. C. Sholander: Multi-variable extended mean values. J. Math. Anal. Appl. *104*, 390–407 (1984).
8 D. H. Lehmer: On the compounding of certain means. J. Math. Anal. Appl. *36*, 183–200 (1971).
9 D. S. Mitrinovič: Analytic Inequalitites. Springer Verlag. Berlin 1970.
10 K. B. Stolarsky: Generalizations of the logarithmic mean. Math. Mag. *48*, 87–92 (1975).
11 E. M. Wright: A generalization of Schur's inequality. Math. Gaz. *40*, 217 (1956).

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Kleine Mitteilung

Sums of a certain family of series

By identifying the sum

$$S_n = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{1}{2^k(n+k+1)} \quad (n \in \mathcal{N} = \{1, 2, 3, \dots\}) \tag{1}$$

with the integral

$$S_n = \int_0^1 \left(1 - \frac{t}{2}\right)^{n-1} t^n \, dt, \tag{2}$$

and evaluating this Eulerian integral, M. Vowe and H.-J. Seiffert [3] have recently shown that

$$S_n = \frac{2^n (n-1)!\, n!}{(2n)!} - \frac{2^{-n}}{n} \quad (n \in \mathcal{N}). \tag{3}$$

In our attempt to find the sum in (1), *without* evaluating the integral in (2), we are led naturally to the fact that the formula (3) is just one of numerous interesting (and useful) consequences of a known result in the theory of the Gaussian hypergeometric series

$$F(a, b; c; z) = 1 + \frac{a\,b}{c}\,\frac{z}{1!} + \frac{a(a+1)\,b(b+1)}{c(c+1)}\,\frac{z^2}{2!} + \ldots, \tag{4}$$

which, for $a = 1$ and $b = c$ (or, alternatively, for $a = c$ and $b = 1$), reduces immediately to the familiar geometric series. In one of his 1836 memoirs [1], Ernst Eduard Kummer (1810–1893) proved the summation theorem [1, p. 134, Theorem 3]:

$$F(a, 1-a; c; \tfrac{1}{2}) = \frac{\Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{c+1}{2}\right)}{\Gamma\left(\frac{c+a}{2}\right)\Gamma\left(\frac{c-a+1}{2}\right)} \qquad (c \neq 0, -1, -2, \ldots), \tag{5}$$

where, as usual, $\Gamma(z)$ denotes the familiar Gamma function satisfying the relationships:

$$\begin{cases} \Gamma(z+1) = z\,\Gamma(z), \ \sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\,\Gamma(z+1/2), \\ \Gamma(n+1) = n! \quad (n \in \mathcal{N} \cup \{0\}), \Gamma(1/2) = \sqrt{\pi} \end{cases} \tag{6}$$

(see also Srivastava and Karlsson [2, pp. 18–19]).
From the definition

$$\binom{\lambda}{0} = 1; \quad \binom{\lambda}{k} = \frac{\lambda(\lambda-1)\ldots(\lambda-k+1)}{k!} \qquad (k \in \mathcal{N}), \tag{7}$$

for an arbitrary (real or complex) λ, it follows readily that

$$\binom{\lambda+k-1}{k} = \frac{\lambda(\lambda+1)\ldots(\lambda+k-1)}{k!} \qquad (k \in \mathcal{N} \cup \{0\}). \tag{8}$$

Making use of (8), and the second relationship in (6), it is fairly easy to state Kummer's summation theorem (5) in the (more relevant) form:

$$S_{\lambda,\mu} \equiv \sum_{k=0}^{\infty} (-1)^k \binom{\lambda-1}{k} \frac{\binom{\lambda+k-1}{k}}{2^k \binom{\mu+k-1}{k}} = \frac{2^{1-\mu}\sqrt{\pi}\,\Gamma(\mu)}{\Gamma\left(\frac{\mu+\lambda}{2}\right)\Gamma\left(\frac{\mu-\lambda+1}{2}\right)} \tag{9}$$

$$(\mu \neq 0, -1, -2, \ldots).$$

Since

$$\binom{n-1}{k} = 0, \qquad k = n, n+1, n+2, \ldots, \tag{10}$$

the sum in (9) would terminate at $k=n-1$ in the special case when $\lambda=n\in\mathcal{N}$. In particular, we have

$$S_{n,n}\equiv\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}2^{-k}=2^{1-n}\qquad(n\in\mathcal{N}).\tag{11}$$

$$S_{n,n+1}\equiv\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{1}{2^k(n+k)}=\frac{2^n(n-1)!\,n!}{(2n)!}\qquad(n\in\mathcal{N}),\tag{12}$$

and

$$S_{n,n+2}\equiv\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{1}{2^k(n+k)(n+k+1)}=\frac{2^{-n}}{n}\qquad(n\in\mathcal{N}).\tag{13}$$

Formula (11) is an obvious consequence of the familiar binomial theorem:

$$\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}=(a+b)^n\qquad(n\in\mathcal{N}\cup\{0\}),\tag{14}$$

or, more generally,

$$\sum_{k=0}^{\infty}\binom{\lambda}{k}z^k=(1+z)^\lambda\qquad(|z|<1;\ \lambda\text{ arbitrary}),\tag{15}$$

which incidentally is related to (4) with $a=-\lambda$, $b=c$, and z replaced by $-z$.
Formulas (12) and (13), together, yield

$$S_n\equiv S_{n,n+1}-S_{n,n+2}=\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{1}{2^k(n+k+1)}\tag{16}$$

$$=\frac{2^n(n-1)!\,n!}{(2n)!}-\frac{2^{-n}}{n}\qquad(n\in\mathcal{N}),$$

which is precisely the summation formula (3) given by Vowe and Seiffert [3].
It is not difficult to deduce from (9) the following generalization of (3):

$$\sum_{k=0}^{\infty}(-1)^k\binom{\lambda-1}{k}\frac{1}{2^k(\lambda+k+1)}=\frac{2^\lambda\Gamma(\lambda)\,\Gamma(\lambda+1)}{\Gamma(2\lambda+1)}-\frac{2^{-\lambda}}{\lambda}\qquad(\lambda\neq 0,-1,-2,\ldots),\tag{17}$$

which holds true for an essentially arbitrary (real or complex) λ.

Some further consequences of the general result (9) are worthy of note. Indeed, for every non-negative integer l, we obtain

$$S_{\lambda,\lambda+2l} \equiv \sum_{k=0}^{\infty} (-1)^k \binom{\lambda-1}{k} \frac{1}{2^k \prod_{j=1}^{2l} (\lambda+k+j-1)} \tag{18}$$

$$= \frac{2^{1-\lambda}\, l!}{(2l)! \prod_{j=1}^{l} (\lambda+j-1)} \quad (\lambda \neq 0, -1, -2, \ldots)$$

and

$$S_{\lambda,\lambda+2l+1} \equiv \sum_{k=0}^{\infty} (-1)^k \binom{\lambda-1}{k} \frac{1}{2^k \prod_{j=0}^{2l} (\lambda+k+j)} \tag{19}$$

$$= \frac{2^{\lambda}\,\Gamma(\lambda)\,\Gamma(\lambda+l+1)}{l!\,\Gamma(2\lambda+2l+1)} \quad (\lambda \neq 0, -1, -2, \ldots),$$

where, as usual, an empty product is to be interpreted as 1.
Upon subtracting (18) from (19) with l replaced by $l-1$, we find that

$$\sum_{k=0}^{\infty} (-1)^k \binom{\lambda-1}{k} \frac{\lambda+k+2l-2}{2^k \prod_{j=1}^{2l} (\lambda+k+j-1)} \tag{20}$$

$$= \frac{2^{\lambda}\,\Gamma(\lambda)\,\Gamma(\lambda+l)}{(l-1)!\,\Gamma(2\lambda+2l-1)} - \frac{2^{1-\lambda}\, l!}{(2l)! \prod_{j=1}^{l} (\lambda+j-1)} \quad (l \in \mathcal{N}),$$

which evidently yields (17) when $l = 1$.
Each of the summation formulas (18), (19), and (20) would terminate, by virtue of (10), in its special case when $\lambda = n \in \mathcal{N}$. Formula (20) thus yields

$$\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{n+k+2l-2}{2^k \prod_{j=1}^{2l} (n+k+j-1)} \tag{21}$$

$$= \frac{2^n (n-1)!\,(n+l-1)!}{(l-1)!\,(2n+2l-2)!} - \frac{2^{1-n}\, l!}{(2l)! \prod_{j=1}^{l} (n+j-1)} \quad (n, l \in \mathcal{N}),$$

which provides us with yet another generalization of the summation formula (3).

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REFERENCES

1 E. E. Kummer: Über die hypergeometrische Reihe.... J. Reine Angew. Math. *15*, 39–83 and 127–172 (1836).
2 H. M. Srivastava, P. W. Karlsson: Multiple Gaussian Hypergeometric Series. Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto 1985.
3 M. Vowe, H.-J. Seiffert: Aufgabe 946. Elem. Math. *42*, 111–112 (1987).

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Aufgaben

Aufgabe 957. Man beweise die Ungleichung

$$\sin(x/2) + \cos x < (\pi - x)/2; \quad 0 < x < \pi.$$

P. Ivady, Budapest, Ungarn

Lösung. Für $0 < x < \pi$ gilt nach einer bekannten Identität

$$\begin{aligned} \sin(x/2) + \cos x &= 2\sin((\pi - x)/4)\cos((3x - \pi)/4) \\ &< 2\sin((\pi - x)/4) \\ &< (\pi - x)/2. \end{aligned}$$

A. A. Jagers, Enschede, NL

Weitere Lösungen sandten S. Arslanagic (Trebinje, YU), A. Bender (Zürich), H. Bopp (Illingen), E. Braune (Linz, A), P. Bracken (Toronto, CD), P. Bundschuh (Köln, BRD), F. Götze (Jena, DDR), M. Hübner (Leipzig, DDR), W. Janous (Innsbruck, A), L. Kuipers (Sierre), Kee-wai Lau (Hongkong), I. Merenyi (Berveni, RU), A. Müller (Zürich), P. Müller (Nürnberg, BRD), H.-J. Seiffert (Berlin), Tsen-Pao Shen (München, BRD), H. M. Smid (Amsterdam, NL), M. Vowe (Therwil), R. Wyss (Flumenthal).

Aufgabe 958. Es seien

$$x_n := \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k}\binom{n}{k} \quad \text{und} \quad y_n := \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k}\binom{n-1}{k-1}.$$

Man berechne: $\lim_{n \to \infty} (x_n + \log y_n)$.

H. Alzer, Waldbröl, BRD