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# ELEMENTE DER MATHEMATIK

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# Some inequalities for the triangle

**Inequalities involving the average side, the average altitude, and the diameter of the circumscribed circle of a triangle**

## I. Introduction

Let $a$, $b$, $c$ be the sides of a triangle inscribed in a circle of diameter $D$. Denote by $l$ and $h$ the averages of the sides $l = (a + b + c)/3$ and the altitudes $h = (h_a + h_b + h_c)/3$, respectively. The quotients $h/l$ and $D/l$ satisfy the following sharp inequalities (with equality for the equilateral triangle):

$$0 < h/l \leq \sqrt{3}/2 \tag{1}$$

$$2/\sqrt{3} \leq D/l < +\infty . \tag{2}$$

The first inequality is due to Santaló [S] and the second to Nakajima [N] and Padoa [P] (see also [B], 6.1 and 5.3). Inequality (1) was also proposed as a Monthly problem (E 1427, page 692, vol 67, 1960; solution on page 296, vol 68, 1961). Apparently none of the 19 solutions received mentioned [S], which had appeared 17 years earlier.
Averaging the inequalities in (1) and (2) gives

$$1/\sqrt{3} \leq \tfrac{1}{2} h/l + \tfrac{1}{2} D/l < +\infty , \tag{3}$$

but the sharp lower bound 1 is not hard to obtain (see [C]). The objective of this paper is to study such inequalities.
Two questions arise naturally: (a) what happens when other convex combinations of $h/l$ and $D/l$ are considered; and (b) what happens when only acute triangles are considered? We obtain sharp inequalities for all combined cases of (a) and (b) (in particular we prove the upper bound for (3) in the case of acute triangles conjectured in [C]).
More precisely, for $0 < \theta < 1$ denote by $g_\theta$ and $G_\theta$ the infimum and supremum of $E_\theta = \theta\, h/l + (1 - \theta)\, D/l$ taken over all triangles and by $g'_\theta$ and $G'_\theta$ the infimum and supremum taken over all *acute* triangles. Our main result is the following theorem whose proof is given below.

**4. Theorem.** *With $\theta_4$ and $\theta_{10}$ given below, we have:*

$$\begin{aligned}
&\text{If } \tfrac{3}{4} \le \theta < 1 \quad \text{then} \quad g_\theta = \sqrt{3}\sqrt{\theta(1-\theta)};\\
&\text{if } \tfrac{1}{2} \le \theta \le \tfrac{3}{4} \quad \text{then} \quad g_\theta = \tfrac{3}{2} - \theta;\\
&\text{if } \theta_{10} \le \theta \le \tfrac{1}{2} \quad \text{then} \quad g_\theta = \theta(3u+2)\sqrt{(1-u)/(1+u)}
\end{aligned} \tag{4.1}$$

*where u is the smaller root of*

$$(u^2-1)(2u+3) + 3(1-\theta)/\theta = 0 \quad \text{in} \quad 0 < u < 1;$$

$$\text{if } 0 < \theta \le \theta_{10} \quad \text{then} \quad g_\theta = (\sqrt{3}/6)(4-\theta).$$

*Only when $0 < \theta < \frac{1}{2}$ is $g_\theta$ attained (at the equilateral triangle for $0 < \theta < \theta_{10}$ and at the isosceles triangle with angle opposite the base equal to $\pi - 2\arccos(u)$ for $\theta_{10} \le \theta < \frac{1}{2}$).*

$$\begin{aligned}
&\text{If } \tfrac{3}{4} \le \theta < 1 \quad \text{then} \quad g'_\theta = \tfrac{3}{2} - \theta;\\
&\text{if } 0 < \theta \le \tfrac{3}{4} \quad \text{then} \quad g'_\theta = g_\theta.
\end{aligned} \tag{4.2}$$

*It is attained only for $0 < \theta < \frac{1}{2}$.*

$$\begin{aligned}
&\text{If } \theta_4 \le \theta < 1 \quad \text{then} \quad G'_\theta = \frac{\sqrt{3}}{6}(4-\theta);\\
&\text{if } \tfrac{3}{23}(5-\sqrt{2}) \le \theta \le \theta_4 \quad \text{then} \quad G'_\theta = 3(-1+\sqrt{2}) + \tfrac{1}{2}(9-7\sqrt{2})\theta;\\
&\text{if } 0 < \theta \le \tfrac{3}{23}(5-\sqrt{2}) \quad \text{then} \quad G'_\theta = \tfrac{3}{2} - \theta.
\end{aligned} \tag{4.3}$$

*Only when $\frac{3}{23}(5-\sqrt{2}) \le \theta \le 1$ is the supremum attained (at the equilateral triangle for $\theta_4 \le \theta < 1$ and at the right isosceles triangle for $\frac{3}{23}(5-\sqrt{2}) \le \theta \le \theta_4$).*

$$G_\theta = +\infty \quad \text{for} \quad 0 < \theta < 1. \tag{4.4}$$

*In the above*

$$\theta_4 = (1/382)(508 + 153\sqrt{2} - 138\sqrt{3} - 113\sqrt{6}) = 0.5460,$$

$$\theta_{10} = 3/(3 + K_{10}) = 0.4799,$$

*where $K_{10} = 3 + 2u_0 - 3u_0^2 - 2u_0^3 = 3.2512$, with*

$$A = \tfrac{1}{12}(22{,}429 + 243\sqrt{5793})^{1/3} + \tfrac{1}{12}(22{,}429 - 243\sqrt{5793})^{1/3} + \tfrac{19}{48}$$

*and*

$$u_0 = \frac{1}{2}\left(\sqrt{A} + \sqrt{\frac{297}{32\sqrt{A}} + \frac{19}{16} - A}\right) - \frac{11}{8} = 0.1801.$$

We close this section with some particularly attractive inequalities obtained from Theorem 4 by assigning special values to $\theta$. In all cases the upper bounds hold for acute triangles, and they are sharp. The lower bounds hold for all triangles, and they are sharp for both acute or arbitrary triangles.

$$2l \le h + D \le (1/2)(-3 + 5\sqrt{2}) \cdot l$$

$$(5/2)\sqrt{3} \cdot l \le h + 3D \le 5 \cdot l$$

$$3l \le 3h + D \le (13/6)\sqrt{3} \cdot l$$

$$7l \le 5h + 3D \le (9/2)\sqrt{3} \cdot l$$

$$(\sqrt{3}/6)(7 + 4\sqrt{2}) \cdot l \le h + (1 + \sqrt{2})D \le (1/2)(4 + 3\sqrt{2}) \cdot l$$

$$(1/6)\sqrt{3}(-3 + 7\sqrt{2}) \cdot l \le (\sqrt{2} - 1)h + \sqrt{2}D \le (1/2)(-1 + 4\sqrt{2}) \cdot l$$

$$405\sqrt{5/7} \cdot l \le 162h + 175D \le (-282 + 444\sqrt{2}) \cdot l$$

$$608\sqrt{7} \cdot l \le 768h + 819D \le (-1305 + 2073\sqrt{2}) \cdot l$$

$$(1725/\sqrt{11}) \cdot l \le 250h + 264D \le (-417 + 667\sqrt{2}) \cdot l.$$

The corresponding values of $\theta$ are: 1/2, 1/4, 3/4, 5/8, $1 - (\sqrt{2}/2)$, $(1/7)(3 - \sqrt{2})$, 162/337, $(16/23)^2$, and 125/257, respectively. Such nice expressions can not be expected for all values of $\theta$, of course. In particular if $\theta = 12/25$ for example (so $K = 13/4$), given any triangle $T$ one can construct with straightedge and compass the number $E_\theta(T)$ but it is impossible to construct the lower bound $g_{12/25}$ or the triangle where it is attained.

## II. Reduction to special cases

Since $h/l$ and $D/l$ are invariant under dilations we will assume in the rest of this paper that $D = 1$. Then if $\alpha$, $\beta$, $\gamma$ are the angles opposite sides $a$, $b$, $c$ we have $a = \sin\alpha$, $b = \sin\beta$, $c = \sin\gamma$, $h_a = bc$, $h_b = ca$, $h_c = ab$. Abbreviate

$$P = a + b + c = \sin\alpha + \sin\beta + \sin\gamma$$

$$Q = ab + bc + ca = \sin\alpha\sin\beta + \sin\beta\sin\gamma + \sin\gamma\sin\alpha \tag{5}$$

and introduce the convenient parameter $K = 3(1 - \theta)/\theta$ (so that $\theta = 3/(3 + K)$); we will frequently use $K$ or a combination of both $K$ and $\theta$ in lieu of $\theta$. Then $E_\theta = \theta(Q + K)/P$ and our problem is to find the extrema of $E_\theta$ as a function of $\alpha$, $\beta$, $\gamma$ subject to

$$\alpha, \beta, \gamma \ge 0, \quad \alpha + \beta + \gamma = \pi \tag{6.a}$$

for the case of all triangles or

$$\tfrac{\pi}{2} \geq \alpha, \beta, \gamma \geq 0, \quad \alpha + \beta + \gamma = \pi \tag{6.b}$$

for acute triangles.

**7. Proposition.** *The interior critical points of $E_\theta$ subject to (6.a) (or (6.b)) satisfy $\alpha = \beta$, $\beta = \gamma$, or $\gamma = \alpha$, i.e., they correspond to isosceles triangles.*

*Proof:* Apply Lagrange multipliers as follows. Denote $f(\alpha, \beta, \gamma) = \alpha + \beta + \gamma - \pi$. Using (5) we get

$$\frac{\partial}{\partial \alpha}(E_\theta + \lambda f) = \frac{\theta}{P^2}\{(P^2 - Q - K)\cos\alpha - P\sin\alpha\cos\alpha + \lambda P^2\}$$

and similar expressions for the other partial derivatives. Thus the simultaneous vanishing of $\partial(E_\theta + \lambda f)/\partial\alpha$, $\partial(E_\theta + \lambda f)/\partial\beta$, and $\partial(E_\theta + \lambda f)/\partial\gamma$, is equivalent to a system of the form

$$L\cos\alpha + M\sin\alpha\cos\alpha + N = 0$$

$$L\cos\beta + M\sin\beta\cos\beta + N = 0$$

$$L\cos\gamma + M\sin\gamma\cos\gamma + N = 0$$

(with $M = P \neq 0$ for interior points, etc.). Therefore the critical points are zeros of

$$\delta(\alpha, \beta, \gamma) = \det\begin{pmatrix} \cos\alpha & \sin\alpha\cos\alpha & 1 \\ \cos\beta & \sin\beta\cos\beta & 1 \\ \cos\gamma & \sin\gamma\cos\gamma & 1 \end{pmatrix}.$$

From the assumptions $0 < \alpha, \beta, \gamma, \alpha + \beta + \gamma = \pi$ one can evaluate $\delta(\alpha, \beta, \gamma)$ (see VI):

$$\delta(\alpha, \beta, \gamma) = 4\sin\frac{\alpha - \beta}{2}\sin\frac{\beta - \gamma}{2}\sin\frac{\gamma - \alpha}{2}\left(1 + 2\sin\frac{\alpha}{2}\cdot\sin\frac{\beta}{2}\cdot\sin\frac{\gamma}{2}\right)$$

and since the last factor cannot vanish (in fact, $\geq 2$ by 2.16 in [B]), two of $\alpha$, $\beta$, $\gamma$ must agree, and the proof of Proposition 7 is complete.
This reduces the calculations to isosceles and degenerate triangles (or isosceles, degenerate and right triangles if we restrict ourselves to acute triangles). In fact interpreting the simplex $\alpha + \beta + \gamma = \pi$, $0 \leq \alpha, \beta, \gamma$ as an equilateral triangle (see Figure 1) we conclude from the proposition above that in order to minimize or maximize $E_\theta$ over all triangles it suffices to do so over the intervals $V_1'' V_1'$ and $V_1'' V_2'$ (the remainder of the boundary simply repeats congruent copies of degenerate triangles already contained in $V_1'' V_2'$). For acute triangles it suffices to consider $RV_1'$ and $RV_2'$, for analogous reasons. Thus from now on only critical points of functions of one variable are considered.

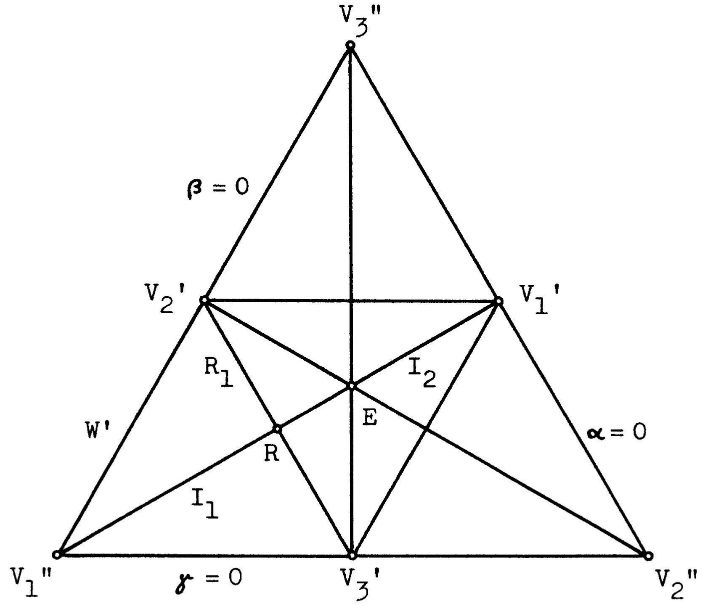


Figure 1.

To avoid repetition we use $V'$ to denote the degenerate triangle with two right angles, and $E$ and $R$ to denote the equilateral and the right isosceles triangles, respectively. In Figure 1 congruent copies of $V'$ appear three times as $V_1'$, $V_2'$ and $V_3'$. We have

$$E_\theta(V') = \tfrac{3}{2} - \theta$$

$$E_\theta(E) = \frac{2\sqrt{3}}{3} - \frac{\sqrt{3}}{6}\theta$$

$$E_\theta(R) = 3(-1 + \sqrt{2}) + \tfrac{1}{2}(9 - 7\sqrt{2})\theta.$$

We will also introduce thirteen constants $K_1, \ldots, K_{13}$ whose values are given in Table 1.

## III. Critical points

In this section we determine the extrema of $E_\theta$ on each of the relevant segments of Figure 1.

**III.1.** Case of $V_1'' V_2'$

These degenerate triangles have $\beta = 0$, $0 \le \gamma \le \pi/2$, $\alpha = \pi - \gamma$ so that

$$E_\theta = \theta(\sin^2\gamma + K)/2\sin\gamma.$$

The critical point $W'$ on $V_1'' V_2'$ is characterized by that value of $\gamma$ at which

$$\frac{dE_\theta}{d\gamma} = \frac{\theta\cos\gamma}{2\sin^2\gamma}(\sin^2\gamma - K) = 0$$

so that $K = \sin^2 \gamma$ is between 0 and 1. Also $E_\theta(W') = \theta\sqrt{K}$, so it is easy to compare the values of $E_\theta$ at the endpoints $V_1''(E_\theta(V_1'') = +\infty)$, $V_2'(E_\theta(V_2') = E_\theta(V'))$ and $E_\theta(W')$.

**III.2.** Case of $RV_2'$

These are right triangles with $\alpha = \pi/2$, $0 \le \beta \le \pi/4$, $\gamma = \pi/2 - \beta$. Setting $y = 1 + \sin\beta + \cos\beta$ we have $E_\theta = y\,\theta/2 + \theta(K-1)/y$ and therefore the only critical points $dE_\theta/d\beta = 0$ satisfy $y^2 = 2(K-1)$ or $dy/d\beta = 0$. Now $dy/d\beta = 0$ corresponds to $R$ and for the triangle $R_1$ corresponding to the other solution we have $E_\theta(R_1) = \theta y$. Since $2 \le y \le 1 + \sqrt{2}$ there is an internal $(0 < \beta < \pi/2)$ critical point only for $K_5 \le K \le K_{13}$. Calculating $dE_\theta/d\beta$ between $\beta = 0\,(V_2')$ and $\beta = \pi/4\,(R)$ we see easily that for $K_5 < K < K_{13}$ we have $dE_\theta/d\beta < 0$ at $\beta = 0$ and $dE_\theta/d\beta > 0$ at $\beta = \pi/4 - \varepsilon$ for $\varepsilon > 0$ sufficiently small. This shows that $E(R_1) \le E(V_2')$ and $E(R_1) \le E(R)$.

**III.3.** Case of $V_1''V_1'$

These are isosceles triangles with $\beta = \gamma$. Introduce the parameter $u = \sqrt{1-b^2} = \sqrt{1-c^2} = \cos\beta = \cos\gamma$ so that $\alpha = 2u\sqrt{1-u^2}$, and

$$E_\theta(a,b,c) = E_\theta(u) = \theta\,\frac{(1+4u)(1-u^2)+K}{2(1+u)\sqrt{1-u^2}}.$$

The triangles $V_1'$, $E$, $R$ and $V_1''$ correspond to $u = 0, 1/2, \sqrt{2}/2$ and 1, respectively. Also abbreviate

$$\eta(x) \quad = (1-x^2)(1+4x),$$

$$\zeta(x) \quad = (1+x)\sqrt{1-x^2},$$

$$\psi(x) \quad = (1-x^2)(3+2x),$$

$$H(x,y) = (\eta(x)+y)/\zeta(x),$$

$$\phi(x) \quad = 2(2+3x)\sqrt{\frac{1-x}{1+x}} = H(x,\psi(x)).$$

Then $E_\theta(u) = (\theta/2)\,H(u,K)$ and

$$\frac{dE_\theta}{du} = \frac{\theta}{2}\,\frac{(1-2u)(\psi(u)-K)}{\zeta(u)(1-u^2)}.$$

Thus the critical points of $E_\theta(u)$ are given by $u = 1/2$ and any roots of

$$\psi(u) = K \tag{8}$$

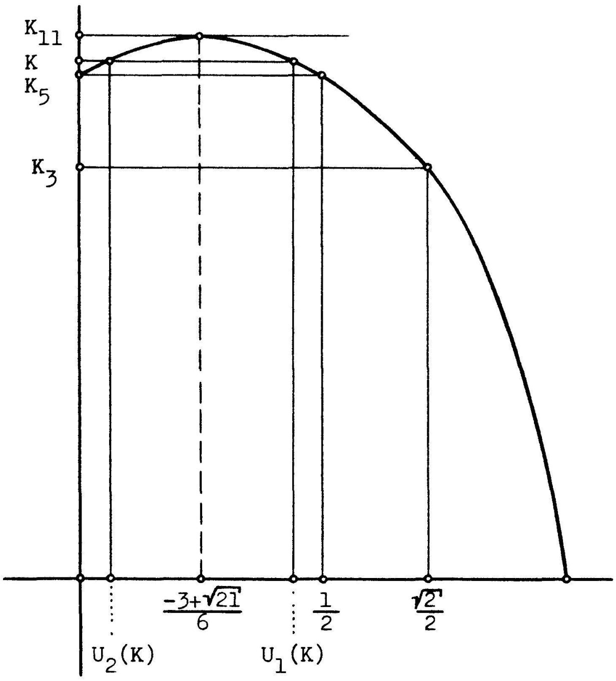

Figure 2.

in the interval $0 \leq u < 1$ for general triangles, $0 \leq u < \sqrt{2}/2$ for acute triangles. (Note that by symmetry the directional derivative of $E_\theta$ perpendicular to $V_1'' V_1'$ vanishes so that every root of (8) is indeed an interior critical point.)
Observing Figure 2 we obtain: For $K > K_{11}$ there are no roots of (8) in $(0,1)$. For $3 = K_5 < K < K_{11}$ there are two roots

$$0 < u_2 = u_2(K) < \tfrac{1}{6}(-3 + \sqrt{21}) < u_1 = u_1(K) < 1/2$$

and for $0 < K < K_5$ there is only one root $1/2 < u_1 < 1$. Furthermore, in the last case, we have $\sqrt{2}/2 > u_1$ (i.e., $u_1$ represents an acute triangle) exactly when $K_3 < K$. We will denote by $I_1$ and $I_2$ the triangles corresponding to $u_1$ and $u_2$, respectively.
From the calculation

$$\psi'(x) = 2(2 + 3x)(1 - x^2)\phi'(x)/\phi(x)$$

and the relation $K = \psi(u_i(K))$ for $i = 1, 2$ we get

$$\begin{aligned}\frac{d}{dK}\log\phi(u_i) &= \frac{\phi'(u_i)}{\phi(u_i)}\frac{du_i}{dK}\\ &= \frac{\psi'(u_i)}{2(2+3u_i)(1-u_i^2)}\frac{du_i}{dK}\\ &= \frac{1}{2(2+3u_i)(1-u_i^2)}\\ &= \frac{1}{2K}\frac{3+2u_i}{2+3u_i}.\end{aligned}$$

We draw two conclusions from this formula. First,

$$\frac{d}{dK}\log\frac{\phi(u_2)}{\phi(u_1)} = \frac{5}{2K}\frac{u_1 - u_2}{(2+3u_1)(2+3u_2)} > 0.$$

Second,

$$\frac{d}{dK}\log E_\theta(I_i) = \frac{d}{dK}\log(\theta\,\phi(u_i))$$

$$= \frac{\theta'}{\theta} + \frac{1}{2K}\frac{3+2u_i}{2+3u_i}$$

$$= -\frac{1}{K+3} + \frac{1}{2K}\frac{3+2u_i}{2+3u_i},$$

which is positive when $K = 3$ for both $u_1 = 1/2$ and $u_2 = 0$. Using $\psi(u_i) = K$, this derivative could only vanish if $u_i$ were a root of $8x^4 + 14x^3 - 5x^2 - 8x + 6 = (2x+3)(4x^3 + x^2 - 4x + 2) = 0$. But this equation has no positive root. Hence from $E_\theta(u_2)/E_\theta(u_1) = \phi(u_2)/\phi(u_2)$ and the foregoing we get:

**III.4.** *$E_\theta(I_1)$, $E_\theta(I_2)$, and the ratio $E_\theta(I_2)/E_\theta(I_1)$, are strictly increasing functions of $K$. Since $E_\theta(I_2) = E_\theta(I_1)$ for $K = K_{11}$ it follows that $E_\theta(I_2) < E_\theta(I_1)$ for $3 < K < K_{11}$.*

## IV. Intersections

According to sections II and III the extrema of $E_\theta$ are achieved for general triangles at one of $E$, $V'$, $V''$, $W'$, $I_1$, $I_2$ and for acute triangles at one of $R$, $E$, $V'$, $R_1$, $I_2$ and $I_1$ (if $I_1$ corresponds to an acute triangle). To determine which points correspond to extrema we consider the seven functions $f^T(K) = E_\theta(T)$ of $K$ where, $T$ is one of $V'$, $W'$, $R$, $R_1$, $E$, $I_1$, $I_2$ and $\theta = 3/(3+K)$. Our strategy is a brute force approach: First we determine the values of $K$ at which each pair of functions $f^T$ and $f^S$ intersect by solving

$$f^T(K) = f^S(K) \tag{9}$$

and then we rank them in the resulting contiguous intervals. All the information is summarized in Figure 3 at the end of this section and Tables 1 and 2.

**IV.1.** $T, S \in \{R, E, V'\}$. These are isosceles triangles with $u_T$ and $u_S$ among $u_R = \sqrt{2}/2$, $u_E = 1/2$, $u_{V'} = 0$ so the intersections occur at the roots of

$$\theta\frac{(1+4u_T)(1-u_T^2)+K}{2(1+u_T)(1-u_T^2)^{1/2}} = \theta\frac{(1+4u_S)(1-u_S^2)+K}{2(1+u_S)(1-u_S^2)^{1/2}}$$

which is linear in $K$. The solutions $K_4$, $K_6$, $K_{12}$ are listed in Table 1.

**IV.2.** $T \in \{I_1, I_2\}$, $S \in \{R, E, V'\}$. Equation (9) reads

$$\frac{\theta}{2}\frac{\eta(u)+\psi(u)}{\zeta(u)} = \theta\frac{(1+4u_S)(1-u_S^2)+K}{2(1+u_S)(1-u_S^2)^{1/2}}$$

where $K = \psi(u)$. Upon squaring this yields a seventh order polynomial in $u$. The roots $r$ in $((-3+\sqrt{21})/6, 1)$ correspond to $T = I_1$ and the roots $s$ in $(0, (-3+\sqrt{21})/6)$ correspond to $T = I_2$.

**IV.2a.** When $S = R$ the equation obtained is

$$(u-1/\sqrt{2})^2(8u^5+8(4+\sqrt{2})u^4+2(19+16\sqrt{2})u^3 + 2(-1+13\sqrt{2})u^2+(1-2\sqrt{2})u+9-4\sqrt{2}) = 0$$

where the fifth order factor has no positive roots and the root $r = 1/\sqrt{2}$ corresponds to the intersection of $f^{I_1}$ and $f^R$ at $K_3$.

**IV.2b.** When $S = E$ the equation is $(u-1/2)^3(8u^4+44u^3+86u^2+33u-9) = 0$ and $r = 1/2 > (-3+\sqrt{21})/6$ gives the intersection of $f^{I_1}$ and $f^E$ at $K_5 = \psi(1/2) = 3$. The quartic factor has one positive root $u_0 = 0.180125573$ whose closed form expression is given in the statement of Theorem 4. It corresponds to the intersection of $f^{I_2}$ and $f^E$ at $K_{10}$.

**IV.2c.** When $S = V'$ the equation obtained is $u^2(4u^5+16u^4+13u^3-27u^2-12u+8) = 0$ and the second factor has two positive roots: $r_1 = 0.961103259$ and $r_2 = 0.407160288$ (both are $> (-3+\sqrt{21})/6$). They correspond to the intersections of $f^{I_1}$ and $f^{V'}$ at $K_1 = \psi(r_1)$ and $K_7 = \psi(r_2)$. The root $u = 0$ corresponds to the intersection of $f^{I_2}(K)$ and $f^{V'}(K)$ at $K_5 = \psi(0) = 3$.

**IV.3.** $T = I_1$, $S = I_2$. As stated in III.4 these intersect only once, at $K_{11}$.

**IV.4.** $T = V'$, $S = R_1$. Equation (9) is $3/2 - 3/(3+K) = (3/(3+K))\sqrt{2K-2}$ so $K = K_5 = 3$.

**IV.5.** $T = E$, $S = R_1$. Recall that $f^{R_1}$ is only defined for $3 \le K \le 5/2 + \sqrt{2}$. The equation (9) is $2\sqrt{3}/3 - (\sqrt{3}/6)(3/(3+K)) = (3/(3+K))\sqrt{2K-2}$, or $16K^2 - 144K + 297 = 0$, and $K_8 = (18 - 3\sqrt{3})/4$ is the only root in the range.

**IV.6.** $T = R$, $S = R_1$. Equation (9) is $3(-1+\sqrt{2}) + \frac{1}{2}(9-7\sqrt{2})(3/(3+K)) = (3/(3+K))\sqrt{2K-2}$ or $K = K_{13} = 5/2 + \sqrt{2}$.

**IV.7.** $T = R_1$, $S \in \{I_1, I_2\}$. The equation is

$$\frac{\theta}{2}\frac{\eta(u)+\psi(u)}{\zeta(u)} = \theta\sqrt{2\psi(u)-2}$$

where $K = \psi(u)$. Thus $u^2(4u^2 + u - 1) = 0$, and the positive root $r = (-1 + \sqrt{17})/8$ corresponds to the intersection of $f^{I_1}$ and $f^{R_1}$ at $K = K_9 = \psi(r) = (135 + 17\sqrt{17})/64$. The root $u = 0$ corresponds to the intersection of $f^{I_2}$ and $f^{R_1}$ at $K_5 = 3 = \psi(0)$.

**IV.8.** $T = V'$, $S = W'$. The equation is $3/2 - 3/(3+K) = (3/(3+K))\sqrt{K}$ so $K = K_2 = 1$.

**IV.9.** $T \in \{E, R\}$, $S = W'$. Equation (9) leads to quadratic equations ($16K^2 - 36K + 81 = 0$ for $T = E$, $4K^2 - 8K + 9 + 4\sqrt{2} = 0$ for $T = R$) with no real roots.

**IV.10.** $T \in \{I_1, I_2\}$, $S = W'$. Equation (9) reads

$$\theta(\eta(u) + \psi(u))/2\zeta(u) = \theta\sqrt{\psi(u)}$$

or $(u-1)(2u^3 - 2u^2 - 4u - 1) = 0$, with no root in $(0,1)$.

**IV.11.** $T = R_1$, $S = W'$. The corresponding functions have disjoint domains so there is no intersection.
The proof of Theorem 4 results now from inspections of Tables 1, 2, 3, and 4.

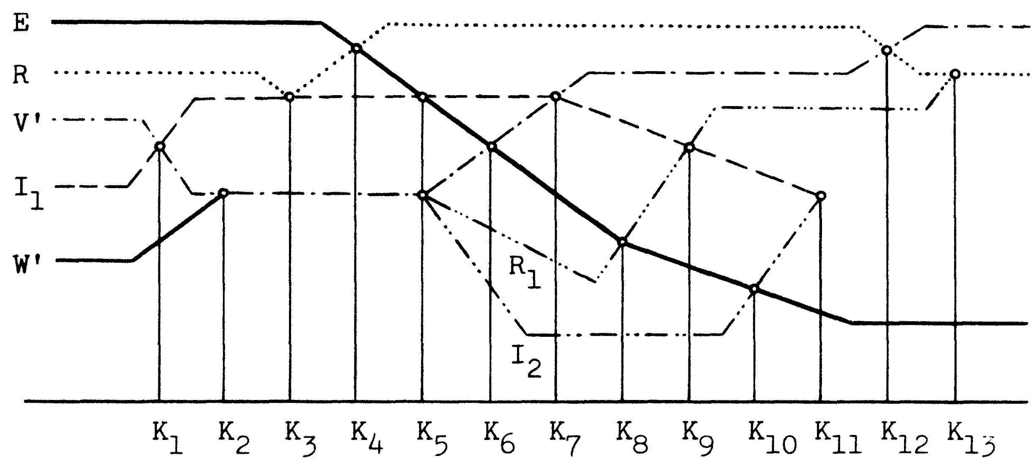


Figure 3. Abstract sketch of the ranks and intersections of $f^T(K)$.

## V. Miscellaneous remarks

**V.1.** In Theorem 4 we gave a "closed form" radical expression for $K_{10}$. This is possible for all $K_i$ except $K_1$ and $K_7$. In fact the polynomial $\tau(u) = 4u^5 + 16u^4 + 13u^3 - 27u^2 - 12u + 8$ is irreducible over the rationals and has precisely two non-real roots. Thus its Galois group is $S_5$ (the symmetric group on five elements) (see [K], Theorem 29). If $K = K_1$ or $K_7$, then for some root $u$ of $\tau$, $K = \psi(u) \in Q(u) \setminus Q$. As $[Q(u):Q] = 5$, we have $Q(K) = Q(u)$. Thus the Galois group of the minimal polynomial for $K$ is $S_5$ as well ($= Aut_Q N$, where N is the normal closure of $Q(K)$ over Q). Hence $K$ can not lie in a radical extension of Q (by Theorem 28 in [K]).

Table 1. Values of $K_i$ and $\theta_i$.

| | | |
|---|---|---|
| $K_0 = 0.0000$ | $= 0$ | $\theta_0 = 1.0000$ |
| $K_1 = 0.3755$ | $= \psi(u)$ [1*] where $u$ is the *larger* root of $r(u)$ [2*] in $(0,1)$ $(u = 0.9611)$ | $\theta_1 = 0.8888$ |
| $K_2 = 1.0000$ | $= 1$ | $\theta_2 = 0.7500$ |
| $K_3 = 2.2071$ | $= (3 + \sqrt{2})/2$ | $\theta_3 = 0.5761$ |
| $K_4 = 2.4948$ | $= (3/194)(39 - 18\sqrt{2} + 33\sqrt{3} + 37\sqrt{6})$ | $\theta_4 = 0.5460$ |
| $K_5 = 3.0000$ | $= 3$ | $\theta_5 = 0.5000$ |
| $K_6 = 3.1801$ | $= (3/11)(3 + 5\sqrt{3})$ | $\theta_6 = 0.4854$ |
| $K_7 = 3.1820$ | $= \psi(u)$ where $u$ is the *smaller* root of $r(u)$ in $(0,1)$ $(u = 0.4072)$ | $\theta_7 = 0.4853$ |
| $K_8 = 3.2010$ | $= (1/4)(18 - 3\sqrt{3})$ | $\theta_8 = 0.4838$ |
| $K_9 = 3.2046$ | $= (1/64)(135 + 17\sqrt{17})$ | $\theta_9 = 0.4835$ |
| $K_{10} = 3.2512$ | $= \psi(u_0)$ where $u_0$ is the root of $\sigma(u)$ [3*] in $(0,1)$ $(u_0 = 0.1801$ [4*]) | $\theta_{10} = 0.4799$ |
| $K_{11} = 3.2821$ | $= (1/18)(27 + 7\sqrt{21})$ | $\theta_{11} = 0.4775$ |
| $K_{12} = 3.4142$ | $= 2 + \sqrt{2}$ | $\theta_{12} = 0.4677$ |
| $K_{13} = 3.9142$ | $= (1/2)(5 + 2\sqrt{2})$ | $\theta_{13} = 0.4339$ |
| $K_\infty = \infty$ | | $\theta_\infty = 0$ |

[1*] $\psi(u) = -2u^3 - 3u^2 + 2u + 3$,
[2*] $r(u) = 4u^5 + 16u^4 + 13u^3 - 27u^2 - 12u + 8$,
[3*] $\sigma(u) = 8u^4 + 44u^3 + 86u^2 + 33u - 9$,
[4*] the exact value of $u_0$ is given in Theorem 4.

Table 2. Values of $K$ at which $f^T(K)$ and $f^S(K)$ intersect. Table entries are the subscripts from Table 1 (e.g., $f^{I_1}$ and $f^{V'}$ intersect at $K = K_1$ and $K = K_7$).

| | $V'$ | $I_2$ | $I_1$ | $E$ | $R$ | $R_1$ | $W'$ |
|---|---|---|---|---|---|---|---|
| $V'$ | – | 5 | 1.7 | 6 | 12 | 5 | 2 |
| $I_2$ | – | – | 11 | 10 | [1*] | 5 | [1*] |
| $I_1$ | – | – | – | 5 | 3 [2*] | 9 | [1*] |
| $E$ | – | – | – | – | 4 | 8 | [1*] |
| $R$ | – | – | – | – | – | 13 | [1*] |
| $R_1$ | – | – | – | – | – | – | [1*] |
| $W'$ | – | – | – | – | – | – | – |

[1*] no intersection,
[2*] intersect, do not cross.

Table 3. (All triangles.) Orderings of $E_\theta(T)$, for $T \in \{W', V', I_1, E_2, E\}$ and $K_i < K < K_j$. The critical point of lowest $E_\theta$ appears on the left; $E_\theta$ increases to the right.

| $i,j$ | | | | |
|---|---|---|---|---|
| 0, 1 | $W'$ | $I_1$ | $V'$ | $E$ |
| 1, 2 | $W'$ | $V'$ | $I_1$ | $E$ |
| 2, 5 | | $V'$ | $I_1$ | $E$ |
| 5, 6 | $I_2$ | $V'$ | $E$ | $I_1$ |
| 6, 7 | $I_2$ | $E$ | $V'$ | $I_1$ |
| 7, 10 | $I_2$ | $E$ | $I_1$ | $V'$ |
| 10, 11 | $E$ | $I_2$ | $I_1$ | $V'$ |
| 11, $\infty$ | $E$ | | | $V'$ |

Table 4. (Acute triangles.) Orderings of $E_\theta(T)$ for $T \in \{I_1, I_2, R, R_1, E, V'\}$ for $K_i < K < K_j$. The critical point with smallest value of $E_\theta$ appears on the left; $E_\theta$ increases to the right. (Remark: $I_1$ is obtuse for $K < K_3$).

| $i, j$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0, 3 | | | $V'$ | | $R$ | $E$ | |
| 3, 4 | | | $V'$ | $I_1$ | $R$ | $E$ | |
| 4, 5 | | | $V'$ | $I_1$ | $E$ | $R$ | |
| 5, 6 | $I_2$ | $R_1$ | $V'$ | $E$ | $I_1$ | $R$ | |
| 6, 7 | $I_2$ | $R_1$ | $E$ | $V'$ | $I_1$ | $R$ | |
| 7, 8 | $I_2$ | $R_1$ | $E$ | $I_1$ | $V'$ | $R$ | |
| 8, 9 | $I_2$ | $E$ | $R_1$ | $I_1$ | $V'$ | $R$ | |
| 9, 10 | | $I_2$ | $E$ | $I_1$ | $R_1$ | $V'$ | $R$ |
| 10, 11 | | $E$ | $I_2$ | $I_1$ | $R_1$ | $V'$ | $R$ |
| 11, 12 | | $E$ | | | $R_1$ | $V'$ | $R$ |
| 12, 13 | | $E$ | | | $R_1$ | $R$ | $V'$ |
| 13, $\infty$ | | $E$ | | | | $R$ | $V'$ |

Setting $\sigma(u) = 8u^4 + 44u^3 + 86u^2 + 33u - 9$, note that the cubic resolvent of $\sigma$ is the irreducible cubic with two non-real roots

$$(4y)^3 - 43(4y)^2 + 435(4y) - 2007$$

with Galois group $S_3$ (the symmetric group on three elements) (see [K], Theorem 29). The Galois group of $\sigma (= Aut_{\mathrm{Q}}\mathrm{N}$ where N is the normal closure of $\mathrm{Q}(u_0)$ where $\sigma(u_0) = 0)$ is $S_4$ (by Theorem 43 in [K]). Since $K_{10} = \psi(u_0) \in \mathrm{Q}(u_0) \backslash \mathrm{Q}$, $\mathrm{Q}(\theta_{10}) = \mathrm{Q}(K_{10}) = \mathrm{Q}(u_0)$ ([K], Ex. 5, p. 53).
Therefore

i) $\theta_{10}$, $K_{10}$ and $u_0$ are of degree 4 over Q ($\sigma$ is irreducible!).
ii) The minimal polynomials of $\theta_{10}$, $K_{10}$ and $u_0$ all have Galois group $S_4$.
iii) None of $\theta_{10}$, $K_{10}$, $u_0$ are constructible with straightedge and compass ([K], p. 195, remark at page bottom).

Thus none of the following are constructible with straightedge and compass (over Q):

i) The minimum relative variation of $E_\theta(T)$ over acute triangles (see V.3):

$$G'_{\theta_{10}}/g'_{\theta_{10}}$$

ii) The upper (lower) bound of $E_\theta(T)$ at the point of minimum relative variation:

$$G'_{\theta_{10}}, g'_{\theta_{10}}.$$

iii) The non-equilateral, isosceles triangle with minimum value of $E_\theta$ at the point of minimum relative variation (angle opposite the base equal to $\pi - 2\arccos u_0$).
iv) The relative weights for minimum relative variation in $E_\theta$: $\theta_{10}$ and $1 - \theta_{10}$.

All of the above are of course constructible given $\theta_{10}$ (or $K_{10}$). For completeness we give the minimal polynomial of $\theta_{10}$ over Q:

$$4871\,\theta^4 - 5939\,\theta^3 + 1356\,\theta^2 + 112\,\theta + 32.$$

**V.2.** The conjecture in [C] that for any acute triangle $T$ we have $E_{1/2}(V') \le E_{1/2}(T) \le E_{1/2}(R)$ follows from Theorem 4. Observe that it can be written as

$$l \le \tfrac{1}{2}(h + D) \le \frac{-3 + 5\sqrt{2}}{4} \cdot l$$

where $E_{1/2}(R) = (-3 + 5\sqrt{2})/4 = 1.017766953$, *i.e.: in any acute triangle the average of the mean altitude and the circumdiameter is within 2% of the mean side.*

**V.3.** From Theorem 4 and III.4 we see that $G'_\theta$, $g'_\theta$ and $g_\theta$ are decreasing functions of $\theta$, so that $\sqrt{3}/2 \le G'_\theta \le 3/2$, $1/2 \le g'_\theta \le 2/\sqrt{3}$ and $0 < g_\theta \le 2/\sqrt{3}$ where the lower limits correspond to $\theta = 1$ and the upper limits to $\theta = 0$. It also follows from Theorem 4 that $G'_\theta/g'_\theta$ is a decreasing function of $K$ for $0 < K \le K_5$ and increasing for $K \ge K_{10}$. Over the interval $K_5 \le K \le K_{10}$, we have $G'_\theta/g'_\theta = E_\theta(R)/E_\theta(I_2) = 2\,E_\theta(R)/\theta\,\phi(u_2(K))$ where $\phi$ and $u_2$ are as in III.3. The logarithmic derivative

$$\frac{d}{dK}\log\frac{G'_\theta}{g'_\theta} = \frac{2(-1 + \sqrt{2})}{3 - \sqrt{2} + 2(-1 + \sqrt{2})K} - \frac{3 + 2u_2(K)}{2K(3u_2(K) + 2)}$$

can only vanish when (using $K = \psi(u_2)$):

$$(u_2 + 3/2)(u_2 - 1/\sqrt{2})(8u_2^2 + 2(1 + 2\sqrt{2})u_2 - 4 + \sqrt{2}) = 0.$$

However the smallest positive root of this equation is $(\sqrt{41 - 4\sqrt{2}} - (2\sqrt{2} + 1))/8$ which is greater than $(-3 + \sqrt{21})/6$, and therefore it can not be $u_2$. Hence the quotient $G'_\theta/g'_\theta$ is monotone over $K_5 \le K \le K_{10}$ and an evaluation of the logarithmic derivative at $K = K_5$ shows it to be a decreasing function of $K$ there. Thus

$$\text{for}\quad K \le K_{10}\colon\ G'_{\theta_{10}}/g'_{\theta_{10}} \le G'_\theta/g'_\theta \le \sqrt{3},$$
$$\text{for}\quad K \ge K_{10}\colon\ G'_{\theta_{10}}/g'_{\theta_{10}} \le G'_\theta/g'_\theta \le 3\sqrt{3}/4,$$

where the upper limits correspond to $\theta = 1$ and to $\theta = 0$, respectively. The minimum value of $G'_\theta/g'_\theta$ is then achieved at $K = K_{10}$ with value

$$\frac{18(-1 + \sqrt{2}) + 3(9 - 7\sqrt{2})\theta_{10}}{\sqrt{3}(4 - \theta_{10})} = 1.010471349$$

(where $\theta_{10} = 3/(3 + K_{10})$), or using the values of $g'_{\theta_{10}}$ and $G'_{\theta_{10}}$:

$$1.01616367 \cdot l \le \theta_{10}h + (1 - \theta_{10})D \le 1.026804277 \cdot l.$$

**VI. Calculation of Determinant**

Here is the calculation used in **II.** of the determinant

$$\delta = \det\begin{pmatrix} 1 & \cos\alpha & \sin\alpha\cos\alpha \\ 1 & \cos\beta & \sin\beta\cos\beta \\ 1 & \cos\gamma & \sin\gamma\cos\gamma \end{pmatrix}.$$

Set $x = e^{i\alpha}$, $y = e^{i\beta}$, $z = e^{i\gamma}$. Then $x\bar{x} = y\bar{y} = z\bar{z} = 1$ and $xyz = -1$. Also substituting in the determinant and using linearity in each column we get $8i\delta = \delta_1 - \delta_2 + \delta_3 - \delta_4$, where

$$\delta_1 = \det\begin{pmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{pmatrix}$$

$$\delta_2 = \det\begin{pmatrix} 1 & x & \bar{x}^2 \\ 1 & y & \bar{y}^2 \\ 1 & z & \bar{z}^2 \end{pmatrix}$$

$$\delta_3 = \det\begin{pmatrix} 1 & \bar{x} & x^2 \\ 1 & \bar{y} & y^2 \\ 1 & \bar{z} & z^2 \end{pmatrix}$$

$$\delta_4 = \det\begin{pmatrix} 1 & \bar{x} & \bar{x}^2 \\ 1 & \bar{y} & \bar{y}^2 \\ 1 & \bar{z} & \bar{z}^2 \end{pmatrix}.$$

Now $\delta_1$ is a Vandermonde determinant so $\delta_1 = (y-x)(z-y)(z-x)$ and using $\bar{x} = 1/x$, etc., and $xyz = -1$ we get

$$\bar{\delta}_1 = (\bar{y}-\bar{x})(\bar{z}-\bar{y})(\bar{z}-\bar{x}) = \frac{x-y}{xy}\cdot\frac{y-z}{yz}\cdot\frac{x-z}{xz} = -\delta_1$$

so $\delta_1$ is purely imaginary. Since $\delta_4 = \bar{\delta}_1$ we have $\delta_1 - \delta_4 = 2\delta_1$.
On the other hand

$$\delta_2 = (xyz)^2\delta_2 = \det\begin{pmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{pmatrix}$$

and by successive subtraction of columns we find $\delta_2 = (xy+yz+zx)(y-x)(z-x)(z-y)$ $= -(\bar{z}+\bar{x}+\bar{y})\delta_1$. Also $\delta_3 = \bar{\delta}_2$ so $-\delta_2+\delta_3 = 2i\,\mathfrak{I}((\bar{z}+\bar{x}+\bar{y})\delta_1) = 2\delta_1\,\mathfrak{R}(\bar{z}+\bar{x}+\bar{y})$. Thus

$4i\delta = (1 + \Re(\bar{z} + \bar{x} + \bar{y}))\,\delta_1$. Finally observe that

$$y - x = e^{i\beta} - e^{i\alpha} = e^{i(\alpha+\beta)/2}\, 2i \sin\frac{\beta - \alpha}{2}$$

and similar formulas hold for $z - x$ and $z - y$. Hence

$$\delta_1 = e^{i(\alpha+\beta)/2} e^{i(\alpha+\gamma)/2} e^{i(\gamma+\beta)/2} (2i)^3 \sin\frac{\beta-\alpha}{2} \sin\frac{\gamma-\alpha}{2} \sin\frac{\gamma-\beta}{2}$$

$$= 8i \sin\frac{\beta-\alpha}{2} \sin\frac{\gamma-\alpha}{2} \sin\frac{\gamma-\beta}{2},$$

and since $\Re(\bar{z} + \bar{x} + \bar{y}) = \cos\alpha + \cos\beta + \cos\gamma$ we get for $\delta$ the value

$$\delta = 2\sin\frac{\beta-\alpha}{2} \sin\frac{\gamma-\alpha}{2} \sin\frac{\gamma-\beta}{2} (1 + \cos\alpha + \cos\beta + \cos\gamma).$$

According to [B, 2.16 and 2.12] this can also be written as

$$\delta = 4\sin\frac{\beta-\alpha}{2} \sin\frac{\gamma-\alpha}{2} \sin\frac{\gamma-\beta}{2} \left(1 + 2\sin\frac{\alpha}{2} \cdot \sin\frac{\beta}{2} \cdot \sin\frac{\gamma}{2}\right)$$

$$= 4\sin\frac{\beta-\alpha}{2} \sin\frac{\gamma-\alpha}{2} \sin\frac{\gamma-\beta}{2} (1 + r)$$

where $r$ is the inradius of the triangle.

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