

Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 44 (1989)
Heft: 3

Artikel: On cubic polynomials giving many primes
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DOI: <https://doi.org/10.5169/seals-41611>

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also modulo $\frac{1}{2}\pi$ reduziert = 0, beträgt. Es bleiben die Terme längs $A_0 A_3$ (mit dem Diederwinkel $\frac{1}{2}\pi - \alpha$ und der Kantenlänge $l = \int_{2 \sin \alpha}^u 1/z \, dz$ zwischen den gelegten Horosphären), und längs $A_0 A_1$ (mit dem Diederwinkel α und der Länge $\int_1^u 1/z \, dz$ ausserhalb der Horosphären). Die elementare Ermittlung der Integrale führt auf

$$\Psi(L(\alpha)) = (\log u - \log |2 \sin \alpha|) \otimes \left(\frac{\pi}{2} - \alpha\right) + (\log u) \otimes \alpha,$$

also nach Reduzierung mod $\frac{1}{2}\pi$ und Addition auf

$$\Psi(L(\alpha)) = \log |2 \sin \alpha| \otimes \alpha, \quad \text{w.z.b.w.}$$

Hans E. Debrunner, Math. Institut, Universität Bern

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On cubic polynomials giving many primes

1. Introduction

In the following «prime» means positive or negative prime, that is an integer of the sequence ... -7, -5, -3, -2, 2, 3, 5, 7, 11 ...

The question of finding quadratic polynomials giving many primes has been widely investigated (see [2, p. 115–117] and [4, p. 141–143]) and we can cite for example $2n^2 - 199$ (Karst, 1973, see [3]) giving 87 primes for $n = 0, 1, \dots, 99$ and $n^2 + n + 41$ (Euler, 1772) giving 86 primes in the same range.

The purpose of this paper is to exhibit some polynomials $an^3 + bn^2 + cn + d$ giving many primes and to explain the way they were discovered. In the mathematical literature we found only one result about this subject in [1, p. 420]: «Escott found that $x^3 + x^2 + 17$ is a prime for $-14, -13, \dots, +10$ » (In fact this is not true since making $x = -3$ yields the number -1).

Arbitrarily we shall say that

« $p(n) = an^3 + bn^2 + cn + d$ gives many primes»

if and only if:

Card $\{n \in [0, 1, 2, \dots, 99]$ such that $p(n)$ is prime $\} \geq 75$.

2. Method of computation

All calculations were performed with a micro-computer, using PASCAL and assembly language.

First step

We determine all polynomials $p(n) = an^3 + bn^2 + cn + d$ satisfying:

- (i) $a = 1$ or $a = 2$.
- (ii) $-1 \leq b \leq 1$ if $a = 1$;
 $-2 \leq b \leq 3$ if $a = 2$.
- (iii) $|c| \leq 15$.
- (iv) $p(n) \not\equiv 0 \pmod{m}$ ($m = 2, 3, 5, 7$; $n \in \mathbb{N}$).
- (v) $|d| \leq 10\,000$.
- (vi) Card $\{n \in [0, 1, \dots, 99]$ such that $p(n)$ is prime $\} \geq 60$.

Comments

1) Condition (ii) is not a real restriction since replacing n by $n + 1$ transforms $p(n)$ into $an^3 + (b + 3a)n^2 + (c + 2b + 3a)n + (d + a + b + c)$ showing that b can be reduced mod $3a$.

2) Without condition (iv) we should have 5,219,739 polynomials to examine, that is 521,973,900 primality tests to perform. With this condition the amount of computation is reduced by a factor 40 and the percentage of interesting polynomials lost is very low.

Practically, for given a , b , and c , to get $d \in [0, \dots, 209]$ ($210 = 2 \cdot 3 \cdot 5 \cdot 7$) such that $an^3 + bn^2 + cn + d$ is never divisible by 2, 3, 5 or 7 we run the following algorithm:

```

For  $p = 2, 3, 5$  and  $7$  do
  for  $n = 0$  to  $p - 1$  do
    for  $d = 0$  to  $p - 1$  do
      if  $(an^3 + bn^2 + cn + d) \bmod p = 0$ 
        then for  $j = d$  to  $209$  step  $p$  do suppress  $j$ .

```

The convenient values of $d \pmod{210}$ are the remaining $j \in [0, \dots, 209]$.

Example: For $a = 2$, $b = 1$ and $c = -7$, we have only to consider $d = 41, 107, 167, 191 \pmod{210}$.

Second step

For every polynomial provided by step 1:

- We search in the interval $[-100, \dots, 100]$ a sequence of 100 consecutive integers $j, j + 1, \dots, j + 99$ satisfying.
Card $\{n \in [j, j + 1, \dots, j + 99] \text{ such that } p(n) \text{ is prime}\}$ is maximum.
- If the maximum is greater than or equal to 75, changing n into $n + j$ in $p(n)$ yields a new polynomial giving at least 75 primes values for $n = 0, 1, \dots, 99$.
- If the maximum is less than 75, the polynomial is rejected.

Third step

In the list of polynomials we get from step 2, there are pairs of polynomials $(p(n), q(n))$ such that $p(n) \equiv -q(-n - k)$ for some integer k . Geometrically this means that graphs of p and q are symmetric with respect to a vertical axis. For every pair of such polynomials we cancel one of them.

3. Results

From step 1 we get 215 polynomials. After step 2 it remains 34 polynomials and after step 3, 21 polynomials giving at least 75 prime values for $n = 0, 1, \dots, 99$. They are listed below with the number of prime values they have for $0 \leq n < 100, 200, \dots, 500$.

Polynomials from step 1 have also been exploited to search the longest sequences of consecutive integers n such that all corresponding values of $p(n)$ are primes:

- Formulas (14) and (15) give 26 primes respectively for $n = 16, \dots, 41$ and $n = 74, \dots, 99$;
- Formula $|2n^3 - 83n^2 + 1157n - 4999|$ gives 34 distinct primes for $n = 0, \dots, 33$.

No	a	b	c	d	Number of primes given by $an^3 + bn^2 + cn + d$ for				
					$n < 100$	$n < 200$	$n < 300$	$n < 400$	$n < 500$
(1)	1	-220	16 119	-392 723	75	134	179	219	261
(2)	1	-199	13 190	-290 869	75	124	163	206	235
(3)	1	-160	8 547	-142 811	75	130	179	221	264
(4)	1	-159	8 420	-148 153	76	124	164	203	238
(5)	1	-151	7 610	-129 097	76	125	168	204	245
(6)	1	-150	7 493	-124 277	76	128	170	217	250
(7)	1	-137	6 270	-95 203	75	121	168	197	233
(8)	1	-125	5 196	-73 291	79	130	174	212	254
(9)	1	-119	4 718	-71 741	75	118	158	190	224
(10)	1	-114	4 343	-54 829	76	125	171	200	232
(11)	1	-111	4 100	-49 367	76	119	150	199	246
(12)	1	-97	3 126	-32 603	75	115	161	195	235
(13)	1	-96	3 059	-32 563	75	127	162	192	225
(14)	1	-82	2 237	-20 407	75	112	149	183	218
(15)	2	-489	39 847	-1 084 553	75	134	176	222	267
(16)	2	-372	23 050	-476 027	75	128	174	211	239
(17)	2	-292	14 202	-231 551	76	124	160	206	252
(18)	2	-289	13 917	-221 891	75	124	172	210	247
(19)	2	-281	13 157	-204 487	78	129	169	214	250
(20)	2	-280	13 072	-203 857	76	123	168	204	240
(21)	2	-199	6 595	-79 657	79	125	174	217	257

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A remark on the gamma function

According to Problem 188, Part II of G. Pólya and G. Szegő [2] for each integrable function $f(x)$ on $0 \leq x \leq 1$ we have:

$$\lim_{n \rightarrow \infty} \frac{1}{\varphi(n)} \sum_{\substack{k=1 \\ (k,n)=1}}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx,$$