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also modulo $\frac{1}{2}\pi$ reduziert $= 0$, beträgt. Es bleiben die Terme längs $A_0 A_3$ (mit dem Diederwinkel $\frac{1}{2}\pi - \alpha$ und der Kantenlänge $l = \int_{2\sin\alpha}^{u} 1/z\,dz$ zwischen den gelegten Horosphären), und längs $A_0 A_1$ (mit dem Diederwinkel α und der Länge $\int_1^u 1/z\,dz$ ausserhalb der Horosphären). Die elementare Ermittlung der Integrale führt auf

$$\Psi(L(\alpha)) = (\log u - \log|2\sin\alpha|) \otimes (\tfrac{\pi}{2} - \alpha) + (\log u) \otimes \alpha,$$

also nach Reduzierung mod $\frac{1}{2}\pi$ und Addition auf

$$\Psi(L(\alpha)) = \log|2\sin\alpha| \otimes \alpha, \quad \text{w.z.b.w.}$$

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On cubic polynomials giving many primes

1. Introduction

In the following «prime» means positive or negative prime, that is an integer of the sequence $\ldots -7,\ -5,\ -3,\ -2,\ 2,\ 3,\ 5,\ 7,\ 11 \ldots$

The question of finding quadratic polynomials giving many primes has been widely investigated (see [2, p. 115–117] and [4, p. 141–143]) and we can cite for example $2n^2 - 199$ (Karst, 1973, see [3]) giving 87 primes for $n = 0, 1, \ldots, 99$ and $n^2 + n + 41$ (Euler, 1772) giving 86 primes in the same range.

The purpose of this paper is to exhibit some polynomials $an^3 + bn^2 + cn + d$ giving many primes and to explain the way they were discovered. In the mathematical literature we found only one result about this subject in [1, p. 420]: «Escott found that $x^3 + x^2 + 17$ is a prime for $-14, -13, \ldots, +10$» (In fact this is not true since making $x = -3$ yields the number -1).

Arbitrarily we shall say that

«$p(n) = an^3 + bn^2 + cn + d$ gives many primes»

if and only if:

Card $\{n \in [0, 1, 2, \ldots, 99]$ such that $p(n)$ is prime$\} \geq 75$.

2. Method of computation

All calculations were performed with a micro-computer, using PASCAL and assembly language.

First step

We determine all polynomials $p(n) = an^3 + bn^2 + cn + d$ satisfying:

(i) $a = 1$ or $a = 2$.
(ii) $-1 \leq b \leq 1$ if $a = 1$;
$-2 \leq b \leq 3$ if $a = 2$.
(iii) $|c| \leq 15$.
(iv) $p(n) \not\equiv 0 \pmod{m}$ $(m = 2, 3, 5, 7;\ n \in \mathbb{N})$.
(v) $|d| \leq 10\,000$.
(vi) Card $\{n \in [0, 1, \ldots, 99]$ such that $p(n)$ is prime$\} \geq 60$.

Comments

1) Condition (ii) is not a real restriction since replacing n by $n + 1$ transforms $p(n)$ into $an^3 + (b + 3a)n^2 + (c + 2b + 3a)n + (d + a + b + c)$ showing that b can be reduced mod $3a$.

2) Without condition (iv) we should have 5,219,739 polynomials to examine, that is 521,973,900 primality tests to perform. With this condition the amount of computation is reduced by a factor 40 and the percentage of interesting polynomials lost is very low.

Practically, for given a, b, and c, to get $d \in [0,\dots,209]$ $(210 = 2\cdot 3\cdot 5\cdot 7)$ such that $a n^3 + b n^2 + c n + d$ is never divisible by 2, 3, 5 or 7 we run the following algorithm:

```
For  p = 2, 3, 5 and 7  do
     for  n = 0 to p − 1  do
          for  d = 0 to p − 1  do
               if  (a n³ + b n² + c n + d) mod p = 0
               then for j = d to 209 step p do suppress j.
```

The convenient values of $d\,(\mathrm{mod}\,210)$ are the remaining $j \in [0,\dots,209]$.

Example: For $a = 2$, $b = 1$ and $c = -7$, we have only to consider $d = 41$, 107, 167, 191 (mod 210).

Second step

For every polynomial provided by step 1:

- We search in the interval $[-100,\dots,100]$ a sequence of 100 consecutive integers j, $j+1,\dots,j+99$ satisfying.
  Card $\{n \in [j, j+1,\dots,j+99]$ such that $p(n)$ is prime$\}$ is maximum.
- If the maximum is greater than or equal to 75, changing n into $n+j$ in $p(n)$ yields a new polynomial giving at least 75 primes values for $n = 0, 1, \dots, 99$.
- If the maximum is less than 75, the polynomial is rejected.

Third step

In the list of polynomials we get from step 2, there are pairs of polynomials $(p(n), q(n))$ such that $p(n) \equiv -q(-n-k)$ for some integer k. Geometrically this means that graphs of p and q are symmetric with respect to a vertical axis. For every pair of such polynomials we cancel one of them.

3. Results

From step 1 we get 215 polynomials. After step 2 it remains 34 polynomials and after step 3, 21 polynomials giving at least 75 prime values for $n = 0, 1, 99$. They are listed below with the number of prime values they have for $0 \le n < 100$, $200, \dots, 500$.
Polynomials from step 1 have also been exploited to search the longest sequences of consecutive integers n such that all corresponding values of $p(n)$ are primes:

- Formulas (14) and (15) give 26 primes respectively for $n = 16, \dots, 41$ and $n = 74, \dots, 99$;
- Formula $|2n^3 - 83n^2 + 1157n - 4999|$ gives 34 distinct primes for $n = 0, \dots, 33$.

No	a	b	c	d	Number of primes given by an^3+bn^2+cn+d for $n<100$	$n<200$	$n<300$	$n<400$	$n<500$
(1)	1	−220	16119	−392723	75	134	179	219	261
(2)	1	−199	13190	−290869	75	124	163	206	235
(3)	1	−160	8547	−142811	75	130	179	221	264
(4)	1	−159	8420	−148153	76	124	164	203	238
(5)	1	−151	7610	−129097	76	125	168	204	245
(6)	1	−150	7493	−124277	76	128	170	217	250
(7)	1	−137	6270	−95203	75	121	168	197	233
(8)	1	−125	5196	−73291	79	130	174	212	254
(9)	1	−119	4718	−71741	75	118	158	190	224
(10)	1	−114	4343	−54829	76	125	171	200	232
(11)	1	−111	4100	−49367	76	119	150	199	246
(12)	1	−97	3126	−32603	75	115	161	195	235
(13)	1	−96	3059	−32563	75	127	162	192	225
(14)	1	−82	2237	−20407	75	112	149	183	218
(15)	2	−489	39847	−1084553	75	134	176	222	267
(16)	2	−372	23050	−476027	75	128	174	211	239
(17)	2	−292	14202	−231551	76	124	160	206	252
(18)	2	−289	13917	−221891	75	124	172	210	247
(19)	2	−281	13157	−204487	78	129	169	214	250
(20)	2	−280	13072	−203857	76	123	168	204	240
(21)	2	−199	6595	−79657	79	125	174	217	257

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A remark on the gamma function

According to Problem 188, Part II of G. Pólya and G. Szegő [2] for each integrable function $f(x)$ on $0 \leqq x \leqq 1$ we have:

$$\lim_{n\to\infty} \frac{1}{\varphi(n)} \sum_{\substack{k=1\\(k,n)=1}}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx,$$