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A Desarguesian dual for Nagel's middlespoint

1. In a paper published in 1836, C. H. Nagel [4] defines the “middlespoint” (Mittenpunkt) of a given triangle ABC in the following manner:

Let S_A, S_B, S_C be the midpoints of BC, CA, AB respectively and I_a, I_b, I_c the centres of the excircles, then the lines $S_A I_a, S_B I_b, S_C I_c$ concur at M , the middlespoint of ABC ,

see also [1]. The name is probably derived from the fact that the point is constructed using “middles”, namely, *centres* of circles and *midpoints* of line segments. In this paper, we derive a dual (line) for this remarkable, but seemingly little known, point and show how this new line relates to some known geometry of the triangle.

2. Desargues's two-triangle theorem in the plane states that if triangles ABC and $A_1 B_1 C_1$ are perspective from a point L , they are perspective from a line l , i.e. if $AA_1 \cap BB_1 \cap CC_1 = L$, then $(AB \cap A_1 B_1) \cup (BC \cap B_1 C_1) \cup (CA \cap C_1 A_1) = l$. Clearly, the converse of this theorem is also its dual, hence, for purposes of this paper, we refer to L and l as “Desarguesian duals”. Also, we will have occasion to make reference to the special case when the triangle $A_1 B_1 C_1$ is inscribed in ABC , i.e. A_1 is on BC , etc. In this instance, L is called the trilinear pole of l and, dually, l is the trilinear polar of L , see [2].

3. In order to facilitate the arguments, we shall use a system of homogeneous coordinates called “trilinear” or “normal”. In order to avoid a possible confusion with trilinear poles and polars, we shall use the term “normal” throughout. In this system, the coordinates (x, y, z) of a point L in the plane of ABC are proportional to the signed distances d_a, d_b, d_c of L from the sides of the triangle of reference ABC , where, obviously, $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$. The distance d_a , for example, is positive if L and the unit point $I = (1, 1, 1)$, the incentre, are on the same side of $a = BC$ and negative otherwise. For instance, the excentre I_A of the excircle opposite vertex A has coordinates $(-1, 1, 1)$, or its projective equivalent, $(1, -1, -1)$.

We now derive the normal line coordinates of l , the trilinear polar of L , that is, if a line t has equation $ux + vy + wz = 0$ then $t = [u, v, w]$ is its normal representation. Also, for the remainder of the paper, we shall denote A_1 by A_L , etc., thus $A_L = (0, y, z)$, $B_L = (x, 0, z)$, $C_L = (x, y, 0)$ and, consequently, $A_L B_L = [yz, xz, -xy] = \left[\frac{1}{x}, \frac{1}{y}, -\frac{1}{z} \right]$, $xyz \neq 0$. Now $C_L'^{[1*]} = AB \cap A_L B_L = (x, -y, 0)$ and, similarly $B_L' = (-x, 0, z)$, $A_L' = (0, y, -z)$; hence the coordinates of $l = A_L' B_L' C_L'$ readily follow. Since this result does not seem to appear in the available literature, we state it as a proposition.

Proposition 1. If a point $L = (x, y, z)$ is in the plane of, but not incident with, a given triangle of reference ABC then, l , its trilinear polar, has coordinates $\left[\frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right]$.

4. The coordinates of the middlespoint M are also readily obtained. The coordinates of the centroid S are easily seen to be of the form $\left(\frac{1}{\sin \alpha}, \frac{1}{\sin \beta}, \frac{1}{\sin \gamma} \right)$, where α, β, γ are

the measures of the vertex angles at A, B, C respectively, however, since $a = 2R \sin \alpha$, where R is the circumradius of ABC , it is more convenient to write $S = \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$.

Now $I_a S_A = [(b-c), b, -c]$, $I_b S_B = [-a, (c-a), c]$, $I_c S_C = [a, -b, (a-b)]$, consequently, $M = (b+c-a, c+a-b, a+b-c)$. Again, we state this result in the form of a proposition.

Proposition 2. The normal coordinates of the midpoint M of a given triangle ABC are given by $M = (s-a, s-b, s-c)$, where $s = \frac{a+b+c}{2}$.

5. Since the triangles $I_A I_B I_C$ and $S_A S_B S_C$ are perspective from the midpoint M , they are, by Desargues's theorem, perspective from a line m , the Desarguesian dual of M , which we shall call the "middlesline" (mittenlinie) of ABC . Following the procedures above, it is an elementary exercise to show that the coordinates of m are $[a(s-a), b(s-b), c(s-c)]$, the details of which we leave as an exercise for the reader. We now state and prove a related result.

Proposition 3. The middlesline is the trilinear polar of the Gergonne point G of the given triangle ABC .

Proof: Since G_A, G_B, G_C are the points of contact of the incircle with the sides of ABC , $BG_A = s-b$ and $G_A C = s-c$, hence $G_A = (0, c(s-c), b(s-b))$ with similiar expressions for G_B, G_C . It now follows that the coordinates of G are $\left(\frac{1}{a(s-a)}, \frac{1}{b(s-b)}, \frac{1}{c(s-c)}\right)$ and by proposition 1, its trilinear polar is $[a(s-a), b(s-b), c(s-c)] = m$ as claimed.

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NOTE

[1*] The "prime" notation is particularly useful here since C_L and C'_L are related by the harmonic conjugacy involution, see [2].