**Zeitschrift:** Elemente der Mathematik

**Herausgeber:** Schweizerische Mathematische Gesellschaft

**Band:** 44 (1989)

**Heft:** 3

**Artikel:** A Desarguesian dual for Nagel's middlespoint

**Autor:** Eddy, Roland H.

**DOI:** https://doi.org/10.5169/seals-41613

# A Desarguesian dual for Nagel's middlespoint

**1.** In a paper published in 1836, C. H. Nagel [4] defines the "middlespoint" (Mittenpunkt) of a given triangle $ABC$ in the following manner:

*Let $S_A$, $S_B$, $S_C$ be the midpoints of $BC$, $CA$, $AB$ respectively and $I_a$, $I_b$, $I_c$ the centres of the excircles, then the lines $S_A I_a$, $S_B I_b$, $S_C I_c$ concur at $M$, the middlespoint of $ABC$,*

see also [1]. The name is probably derived from the fact that the point is constructed using "middles", namely, *centres* of circles and *midpoints* of line segments. In this paper, we derive a dual (line) for this remarkable, but seemingly little known, point and show how this new line relates to some known geometry of the triangle.

**2.** Desargues's two-triangle theorem in the plane states that if triangles $ABC$ and $A_1B_1C_1$ are perspective from a point $L$, they are perspective from a line $l$, i.e. if $AA_1 \cap BB_1 \cap CC_1 = L$, then $(AB \cap A_1B_1) \cup (BC \cap B_1C_1) \cup (CA \cap C_1A_1) = l$. Clearly, the converse of this theorem is also its dual, hence, for purposes of this paper, we refer to $L$ and $l$ as "Desarguesian duals". Also, we will have occasion to make reference to the special case when the triangle $A_1B_1C_1$ is inscribed in $ABC$, i.e. $A_1$is on $BC$, etc. In this instance, $L$ is called the trilinear pole of $l$ and, dually, $l$ is the trilinear polar of $L$, see [2].

**3.** In order to facilitate the arguments, we shall use a system of homogeneous coordinates called "trilinear" or "normal". In order to avoid a possible confusion with trilinear poles and polars, we shall use the term "normal" throughout. In this system, the coordinates $(x, y, z)$ of a point $L$ in the plane of $ABC$ are proportional to the signed distances $d_a$, $d_b$, $d_c$ of $L$ from the sides of the triangle of reference $ABC$, where, obviously, $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$. The distance $d_a$, for example, is positive if $L$ and the unit point $I = (1, 1, 1)$, the incentre, are on the same side of $a = BC$ and negative otherwise. For instance, the excentre $I_A$ of the excircle opposite vertex $A$ has coordinates $(-1, 1, 1)$, or its projective equivalent, $(1, -1, -1)$.
We now derive the normal line coordinates of $l$, the trilinear polar of $L$, that is, if a line $t$ has equation $ux + vy + wz = 0$ then $t = [u, v, w]$ is its normal representation. Also, for the remainder of the paper, we shall denote $A_1$ by $A_L$, etc., thus $A_L = (0, y, z)$, $B_L = (x, 0, z)$, $C_L = (x, y, 0)$ and, consequently, $A_LB_L = [yz, xz, -xy] = \left[\frac{1}{x}, \frac{1}{y}, -\frac{1}{z}\right]$, $xyz \neq 0$. Now $C_L'^{[1*]} = AB \cap A_LB_L = (x, -y, 0)$ and, similarly $B_L' = (-x, 0, z)$, $A_L' = (0, y, -z)$; hence the coordinates of $l = A_L'B_L'C_L'$ readily follow. Since this result does not seem to appear in the available literature, we state it as a proposition.

**Proposition 1.** If a point $L = (x, y, z)$ is in the plane of, but not incident with, a given triangle of reference $ABC$ then, $l$, its trilinear polar, has coordinates $\left[\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right]$.

**4.** The coordinates of the middlespoint $M$ are also readily obtained. The coordinates of the centroid $S$ are easily seen to be of the form $\left(\frac{1}{\sin\alpha}, \frac{1}{\sin\beta}, \frac{1}{\sin\gamma}\right)$, where $\alpha$, $\beta$, $\gamma$ are the measures of the vertex angles at $A$, $B$, $C$ respectively, however, since $a = 2R \sin\alpha$, where $R$ is the circumradius of $ABC$, it is more convenient to write $S = \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$. Now $I_a S_A = [(b-c), b, -c]$, $I_b S_B = [-a, (c-a), c]$, $I_c S_C = [a, -b, (a-b)]$, consequently, $M = (b+c-a, c+a-b, a+b-c)$. Again, we state this result in the form of a proposition.

**Proposition 2.** The normal coordinates of the middlespoint $M$ of a given triangle $ABC$ are given by $M = (s-a, s-b, s-c)$, where $s = \dfrac{a+b+c}{2}$.

**5.** Since the triangles $I_A I_B I_C$ and $S_A S_B S_C$ are perspective from the middlespoint $M$, they are, by Desargues's theorem, perspective from a line $m$, the Desarguesian dual of $M$, which we shall call the "middlesline" (mittenlinie) of $ABC$. Following the procedures above, it is an elementary exercise to show that the coordinates of $m$ are $[a(s-a), b(s-b), c(s-c)]$, the details of which we leave as an exercise for the reader. We now state and prove a related result.

**Proposition 3.** The middlesline is the trilinear polar of the Gergonne point $G$ of the given triangle $ABC$.
Proof: Since $G_A$, $G_B$, $G_C$ are the points of contact of the incircle with the sides of $ABC$, $BG_A = s-b$ and $G_A C = s-c$, hence $G_A = (0, c(s-c), b(s-b))$ with similiar expressions for $G_B$, $G_C$. It now follows that the coordinates of $G$ are $\left(\dfrac{1}{a(s-a)}, \dfrac{1}{b(s-b)}, \dfrac{1}{c(s-c)}\right)$ and by proposition 1, its trilinear polar is $[a(s-a), b(s-b), c(s-c)] = m$ as claimed.

Roland H. Eddy, Memorial University of Newfoundland,
St. John's, Newfoundland, Canada

REFERENCES

1 Baptist P.: Über Nagelsche Punktepaare, Mathematische Semesterberichte *1*, 118–126 (1988).
2 Coxeter H. S. M.: The Real Projective Plane, 2nd ed., Cambridge University Press, 1961.
3 Johnson R. A.: Advanced Euclidean Geometry (Modern Geometry), Dover, NY (1969).
4 Nagel C. H. v.: Untersuchungen über die wichtigsten zum Dreiecke gehörenden Kreise, Leipzig (1836).
5 Sommerville D. M. Y.: Analytical Conics, G. Bell and Sons Ltd., London (1956).
6 Springer C. E.: Geometry and Analysis of Projective Spaces, W. H. Freeman, San Francisco (1964).

NOTE

[1*] The "prime" notation is particularly useful here since $C_L$ and $C'_L$ are related by the harmonic conjugacy involution, see [2].

 0013-6018/89/030079-02$1.50+0.20/0