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A characterization of the tangent function

Real-valued functions T satisfying the identity $T(u+v)=\dfrac{T(u)+T(v)}{1-T(u)T(v)}$ are studied. It is proved that the tangent function is the only such function having domain $\{x: x \text{ real}, x \neq \frac{\pi}{2}+m\pi, m \text{ an integer}\}$ and satisfying $T'(0)=1$.

During the past decade, several of our articles ([1], [2], [3], [4], [5]) have suggested a more fundamental role for the tangent function, tan, in the curriculum. Many of these suggestions depend on the fact that $\tan(u+v)=\dfrac{\tan u+\tan v}{1-\tan u \tan v}$ whenever the right-hand side is defined. It seems natural to ask whether this functional identity characterizes tan. Accordingly, this note considers the class of *tangential functions*, by which we mean real-valued functions T of a real variable such that $T(u+v)=\dfrac{T(u)+T(v)}{1-T(u)T(v)}$ whenever the right-hand side is defined. In proposition 1(c), we produce infinitely many discontinuous tangential functions, thus answering the above question in the negative. On the other hand, Theorem 3 establishes that tan is the only tangential function T which is defined at all real numbers other than $\frac{\pi}{2}+m\pi$ (for m an integer) and satisfies $T'(0)=1$. We hope that the material in this note will find use as enrichment material in introductory courses on calculus.
We begin by collecting some examples of tangential functions.

Proposition 1. Each of the following functions T is tangential:

(a) $T(x)=1$ for each real number x;
(b) $T(x)=-1$ for each real number x;
(c) Let p be a fixed prime number. If x is a real number, put

$$T(x)=\begin{cases} 0 & \text{if} \quad x=\dfrac{m}{p^n} \text{ for some integers } m \text{ and } n \\ 1 & \text{otherwise.}\end{cases}$$

Proof. (a) and (b): The functional identity holds by default since $1-T(u)T(v)\equiv 0$ means that the identity's right-hand side is never defined.
(c): We shall verify the functional identity. Without loss of generality, $T(u)T(v)\neq 1$. Thus at least one of u, v is of the form $\dfrac{m}{p^n}$. If both u and v have this form, so does $u+v$, in which case the functional identity reduces to the truism $0=\dfrac{0+0}{1-0}$. On the other hand, if only one of u, v has the form $\dfrac{m}{p^n}$, then $u+v$ does not have this form, in which case the identity reduces to either $1=\dfrac{1+0}{1-0}$ or $1=\dfrac{0+1}{1-0}$. ■

A tangential function need not be continuous (and, hence, need not be differentiable).

Indeed, if T is as in Proposition 1(c), then for each real number c, $\lim_{x \to c} T(x)$ does not exist. In Proposition 2(b), (c), we examine what can be said about a tangential function which is continuous (or differentiable).

Proposition 2. Let T be a tangential function. Then:

(a) If $T(0)$ is either 1 or -1, then $T(x) = T(0)$ for each x in the domain of T.
(b) Suppose that the domain of T contains a neighbourhood of 0. If T is continuous at 0, then T is a continuous function.
(c) Suppose that the domain of T contains a neighbourhood of 0 and that $T'(0) = 1$. Then T is a differentiable function. In fact, $T'(x) = 1 + T(x)^2$ for each x in the domain of T. Moreover, $T(0) = 0$ and T is increasing on each subinterval of its domain.

Proof. (a) Suppose $T(0) = \pm 1$ and $T(x) \neq T(0)$. Then $1 - T(x)T(0) \neq 0$, and so the functional identity of tangential functions leads to

$$T(x) = T(x+0) = \frac{T(x) + T(0)}{1 - T(x)T(0)} = \frac{T(x) \pm 1}{1 \mp T(x)},$$

whence $T(x)[1 \mp T(x)] = T(x) \pm 1$ and $T(x)^2 = -1$. This contradicts the fact that T is real-valued
(b) Since constant functions are continuous, (a) allows us to suppose that $T(0) \neq \pm 1$. Hence $1 - T(0)^2 \neq 0$, and so the functional identity gives

$$T(0) = T(0+0) = \frac{T(0) + T(0)}{1 - T(0)^2},$$

whence $T(0)[1 - T(0)^2] = 2T(0)$ and $0 = T(0)^3 + T(0) = T(0)[T(0)^2 + 1]$. As $T(0)^2 + 1 \neq 0$, we have $T(0) = 0$. By hypothesis, $\lim_{h \to 0} T(h) = T(0) = 0$. For each x in the domain of T,

$$\lim_{h \to 0} T(x+h) - T(x) = \lim_{h \to 0} \frac{T(x) + T(h)}{1 - T(x)T(h)} - T(x) = \lim_{h \to 0} \frac{[1 + T(x)^2]\,T(h)}{1 - T(x)T(h)},$$

which, by limit theorems, is just $\frac{[1 + T(x)^2]0}{1 - (T(x))0} = 0$. Thus $\lim_{h \to 0} T(x+h) = T(x)$, and so T is continuous at x, proving (b).
(c) Since $T'(0) \neq 0$, it follows from (a) that $T(0) \neq \pm 1$. Hence, by the proof of (b), we have $T(0) = 0$. It follows that $\lim_{h \to 0} \frac{T(h)}{h} = \lim_{h \to 0} \frac{T(h) - T(0)}{h} = T'(0) = 1$. Now, for each x in the domain of T, we see, as in the proof of (b), that

$$\lim_{h \to 0} \frac{T(x+h) - T(x)}{h} = \lim_{h \to 0} [1 + T(x)^2] \frac{T(h)}{h} \left[\frac{1}{1 - T(x)T(h)}\right]$$

$$= [1 + T(x)^2] \cdot 1 \cdot \left[\frac{1}{1 - T(x) \cdot 0}\right];$$

that is, $T'(x) = 1 + T(x)^2$. In particular, $T'(x) > 0$. The final assertion is a standard consequence of the Mean Value Theorem. ■

We next obtain the desired characterization of tan. For motivation, note that $\tan'(0) = \sec^2(0) = 1^2 = 1$.

Theorem 3. *Let T be a tangential function such that $T'(0) = 1$ and each real number $x \neq \frac{\pi}{2} + m\pi$ (for m an integer) is in the domain of T. Then $T = \tan$.*

Proof. First, we restrict attention to x in the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. By Proposition 2(c), T satisfies the variables-separable differential equation $y' = 1 + y^2$. This leads to

$$x = x - 0 = \int_0^x dt = \int_{T(0)}^{T(x)} \frac{ds}{1+s^2} = \int_0^{T(x)} \frac{ds}{1+s^2}$$

$$= \tan^{-1}(T(x)) - \tan^{-1}(0) = \tan^{-1}(T(x)) - 0 = \tan^{-1}(T(x)).$$

Hence, $T(x) = \tan(\tan^{-1}(T(x))) = \tan(x)$ for all x in $(-\frac{\pi}{2}, \frac{\pi}{2})$. [Remark: A short classroom discussion might well end here, as we have just used/reinforced the fundamental theorem of calculus and the chain rule, in the guise of definite integration by change of variable.]
Next, we focus on $\frac{\pi}{2} < x (\neq \frac{\pi}{2} + m\pi)$. First, suppose $\frac{\pi}{2} < x < \pi$. Then $x = 2u = u + v$, where $u = v$ is in $(\frac{\pi}{4}, \frac{\pi}{2})$. As T and tan are both tangential, we may argue as follows, using the result of the preceding paragraph:

$$T(x) = T(u+v) = \frac{T(u) + T(v)}{1 - T(u)T(v)} = \frac{\tan(u) + \tan(v)}{1 - \tan(u)\tan(v)} = \tan(u+v) = \tan(x).$$

Moreover, Proposition 2(b) yields that T is continuous, and so, since tan is also continuous, we have $T(\pi) = \lim_{x \to \pi^-} T(x) = \lim_{x \to \pi^-} \tan(x) = \tan(\pi) = 0$. Hence, by mathematical induction (and the fact that T is tangential), we have $T(n\pi) = 0$ for each positive integer n. It now follows, by reasoning as above, that T and tan agree on $(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2})$. Indeed, if x is in this interval, then $x - n\pi = \tan^{-1}(T(x)) - \tan^{-1}(T(n\pi))$, whence $T(x) = \tan(x - n\pi) = \tan(x)$. Hence, T and tan agree on $(-\frac{\pi}{2}, \infty) \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\}$.
Next, we focus on $-\frac{\pi}{2} > x (\neq \frac{\pi}{2} + m\pi)$. First, suppose $-\pi < x < -\frac{\pi}{2}$. Then $x = 2u = u + v$, where $u = v$ is in $(-\frac{\pi}{2}, -\frac{\pi}{4})$. In particular, $T(u) = \tan(u)$. It follows via tangentiality as above that $T(x) = T(u+v) = \tan(u+v) = \tan(x)$. Then, by considering the limit as x approaches $-\pi$ from the right and invoking continuity, we see that $T(-\pi) = 0$. By mathematical induction and tangentiality, $T(-n\pi) = 0$ for each positive integer n. It now follows, by reasoning as above, that T and tan agree on $(-n\pi - \frac{\pi}{2}, -n\pi + \frac{\pi}{2})$. Hence, T and tan agree on the domain of tan.
We have seen that $T(x) = \tan(x)$ if $x \neq \frac{\pi}{2} + m\pi$ (for m an integer). To complete the proof, it suffices to show that $T(\frac{\pi}{2} + m\pi)$ is not defined. This, however, is a consequence of the continuity of T, since $\lim_{x \to \frac{\pi}{2} + m\pi} T(x) = \lim_{x \to \frac{\pi}{2} + m\pi} \tan(x)$ does not exist. ■

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Kleine Mitteilungen

An explicit formula about the convex hull of random points

Denote by $V_n^{(d)}$ the expected volume of the convex hull of n points chosen independently according to a given probability measure μ in Euclidean d-space E^d. For $d = 2, 3$ and μ the uniform distribution on a convex body in E^d, Affentranger [1], [2] has shown that

$$V_{d+2m}^{(d)} = \sum_{k=1}^{m} \gamma_k \binom{d+2m}{2k-1} V_{d+2m-2k+1}^{(d)} \quad (m = 1, 2, \ldots), \tag{1}$$

where the γ_k can be obtained recursively from $\gamma_1 = \frac{1}{2}$, $2\gamma_k = 1 - \sum_{i=1}^{k-1} \binom{2k-1}{2i-1} \gamma_i \, (k \geq 2)$.

Recently, Buchta [3] has extended this result to arbitrary dimensions d and to arbitrary probability measures μ on E^d. The key point in [3] is the existence of a moment functional $\mathcal{M}$ such that

$$V_{d+1+n}^{(d)} = \binom{d+1+n}{d+1} \mathcal{M}(x^n + (1-x)^n). \tag{2}$$

(See [4] for the definition of moment functionals.)
In this note we show that in formula (1) the γ_k can be expressed explicitly by

$$\gamma_k = (2^{2k} - 1) \frac{B_{2k}}{k} \quad (k = 1, 2, \ldots). \tag{3}$$

Here the B_n are the *Bernoulli numbers* (see e.g. [5], section 1.13), defined by the generating series $z/(e^z - 1) = \sum_{n=0}^{\infty} B_n z^n/n!$. In our proof of formula (1) we can avoid the elimination process used in [1].