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L'Hôpital's rule, a counterexample

1. Introduction

In a recent article, Boas, [1], showed how to construct counterexamples to L'Hôpital's rule, $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$, when the condition $g'(x) \neq 0$ is not satisfied. Boas also emphasized that the trouble lies in changes of sign of the derivative g', not the mere presence of zeros of g'. The sign changes of g', however, imply the existence of zeros, by the intermediate value property of the derivative. This is not true for one-sided derivatives and in Section 3 we give an example where the right-hand derivative never vanishes but changes sign «too often» so that the rule fails. This counterexample brings out clearly the geometry behind the failure and has the additional advantage that $\lim \frac{f(x)}{g(x)}$ actually exists. A suitable theorem for one-sided derivatives and monotonic g precedes this in Section 2.

2. L'Hôpital rules

Theorem 1. *Let f and g be continuous on (a,b) and suppose that g is monotonic. If*

$$\lim_{x \uparrow b} f(x) = 0 = \lim_{x \uparrow b} g(x)$$

and

$$\lim_{x \uparrow b} \frac{f'_+(x)}{g'_+(x)} = l,$$

then

$$\lim_{x \uparrow b} \frac{f(x)}{g(x)} = l.$$

Proof. We may assume that g is increasing. (If it is decreasing we consider $-g$.) For any $\varepsilon > 0$, the following holds. There exists δ, with $0 < \delta < b - a$, such that, for $x \in (b - \delta, b)$, $f'_+(x)$ and $g'_+(x)$ exist and $\frac{f'_+(x)}{g'_+(x)}$ makes sense and therefore $g'_+(x) \neq 0$ and

$$l - \frac{\varepsilon}{2} < \frac{f'_+(x)}{g'_+(x)} < l + \frac{\varepsilon}{2}.$$

During the rest of the proof x will be restricted to the interval $(b-\delta, b)$. Since g is increasing and $g'_+(x) \neq 0$, $g'_+(x) > 0$. Hence

$$\left(l-\frac{\varepsilon}{2}\right) g'_+(x) < f'_+(x) < \left(l+\frac{\varepsilon}{2}\right) g'_+(x). \tag{1}$$

The functions $\left(l+\frac{\varepsilon}{2}\right) g(x) - f(x)$ and $f(x) - \left(l-\frac{\varepsilon}{2}\right) g(x)$ are monotonic increasing because they are continuous and have by (1) positive right-hand derivatives (see Theorem 1 of [5]). Hence, for any y such that $b-\delta < x < y < b$,

$$\left(l-\frac{\varepsilon}{2}\right)(g(y)-g(x)) < f(y) - f(x) < \left(l+\frac{\varepsilon}{2}\right)(g(y)-g(x)). \tag{2}$$

Letting y approach b from below we obtain

$$-\left(l-\frac{\varepsilon}{2}\right) g(x) \leqq -f(x) \leqq -\left(l+\frac{\varepsilon}{2}\right) g(x).$$

Since $\lim_{x \uparrow b} g(x) = 0$ and g is strictly increasing, $g(x) < 0$. Hence

$$l-\varepsilon < \frac{f(x)}{g(x)} < l+\varepsilon.$$

Remark. Theorems similar to Theorem 1 hold for limits from the right, limits, and also limits at $+\infty$ or $-\infty$. Also, the right-hand derivatives can be replaced by left-hand derivatives in the theorem without affecting its validity. There is also no difficulty in modifying the proof if $l = +\infty$ or $-\infty$.

We now consider the case of the indeterminate form "∞/∞". It is convenient to consider the following special case first.

Theorem 2. *If f is continuous on (a, ∞) and $\lim_{x \to \infty} f'_+(x) = l$, then*

$$\lim_{x \to \infty} \frac{f(x)}{x} = l.$$

Proof. It is sufficient to prove the theorem with $l = 0$. With $l \neq 0$, we could then consider the function $f(x) - lx$.

For all $\varepsilon > 0$ there exists $R > 0$ such that, for $x \geq R$,

$$-\frac{\varepsilon}{2} < f'_+(x) < \frac{\varepsilon}{2}.$$

Hence, using the same argument which was used to derive (2) from (1) in Theorem 1, we have, for $x > R$,

$$-\frac{\varepsilon}{2}(x-R) < f(x) - f(R) < \frac{\varepsilon}{2}(x-R).$$

So for $x > R$,

$$-\frac{\varepsilon}{2} < -\frac{\varepsilon}{2}\left(1-\frac{R}{x}\right) < \frac{f(x)}{x} - \frac{f(R)}{x} < \frac{\varepsilon}{2}\left(1-\frac{R}{x}\right) < \frac{\varepsilon}{2}$$

i.e.

$$-\frac{\varepsilon}{2} + \frac{f(R)}{x} < \frac{f(x)}{x} < \frac{\varepsilon}{2} + \frac{f(R)}{x}.$$

Thus, for $x > \max\left(R, \frac{2|f(R)|}{\varepsilon}\right)$,

$$-\varepsilon < \frac{f(x)}{x} < \varepsilon.$$

Theorem 3. *Let f and g be continuous on* (a, ∞) *and suppose that g is monotonic. If*

$$\lim_{x\to\infty} |g(x)| = +\infty \quad \textit{and} \quad \lim_{x\to\infty} \frac{f'_+(x)}{g'_+(x)} = l,$$

then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = l.$$

We need three prerequisites for the proof. Firstly, the usual rule for differentiation of the inverse function holds for one-sided derivatives; secondly, the chain rule holds for one-sided derivatives if the inner function is strictly increasing; thirdly, for the limit of a composite function we have $\lim_{x\to\infty} F(G(x)) = \lim_{y\to\infty} F(y)$ *if* $\lim_{x\to\infty} G(x) = +\infty$.

Proof. For sufficiently large x, $f'_+(x)$ and $g'_+(x)$ exist and $\frac{f'_+(x)}{g'_+(x)}$ makes sense and therefore $g'_+(x) \neq 0$. Consequently g is strictly monotonic and we may assume that g is strictly increasing because if g is strictly decreasing we consider $-g$. Let $F(x) = f(g^{-1}(x))$. Then

$$F'_+(x) = \frac{f'_+(g^{-1}(x))}{g'_+(g^{-1}(x))}$$

so that

$$\lim_{x\to\infty} F'_+(x) = \lim_{x\to\infty} \frac{f'_+(g^{-1}(x))}{g'_+(g^{-1}(x))}$$

$$= \lim_{y\to\infty} \frac{f'_+(y)}{g'_+(y)}$$

$$= l\,.$$

Hence, by Theorem 2

$$l = \lim_{x\to\infty} \frac{F(x)}{x}$$

$$= \lim_{x\to\infty} \frac{f(g^{-1}(x))}{x}$$

$$= \lim_{y\to\infty} \frac{f(y)}{g(y)}\,.$$

Remark. Theorem 3 also holds if the right-hand derivatives are replaced by left-hand derivatives. Also the proof of Theorem 2 can be modified to allow the cases $l = +\infty$ or $l = -\infty$ so that Theorem 3 holds for these cases as well.

3. A counterexample

In this example g is not monotonic and the conclusion of Theorem 3 is shown to be false. Let $\{x_n\}$ be a strictly increasing sequence of real numbers satisfying

$$\lim_{n\to\infty} x_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} (x_{n+1} - x_n) = 0\,,$$

e.g. $x_n = \sum_{k=1}^{n} \frac{1}{k}$. Let $f(x) = x$ and let g be the function whose graph is the union of line segments which join the pairs of points $(x_{2n-1}, x_{2n-1} - 1)$ and $(x_{2n}, x_{2n} + 1)$, $n = 1, 2, \ldots$ and the pairs of points $(x_{2n}, x_{2n} + 1)$ and $(x_{2n+1}, x_{2n+1} - 1)$, $n = 1, 2, \ldots$ (see figure 1). So

$$g(x) = \begin{cases} x_{2n} + 1 + [1 + 2(x_{2n} - x_{2n-1})^{-1}](x - x_{2n}), & x_{2n-1} \leqq x \leqq x_{2n} \\ x_{2n} + 1 + [1 - 2(x_{2n+1} - x_{2n})^{-1}](x - x_{2n}), & x_{2n} \leqq x \leqq x_{2n+1} \end{cases}.$$

Since $\lim_{x\to\infty} |g'_+(x)| = +\infty$, $\lim_{x\to\infty} \left|\frac{f'_+(x)}{g'_+(x)}\right| = 0$ and therefore $\lim_{x\to\infty} \frac{f'_+(x)}{g'_+(x)} = 0$. However,

$$x - 1 \leqq g(x) \leqq x + 1\,,$$

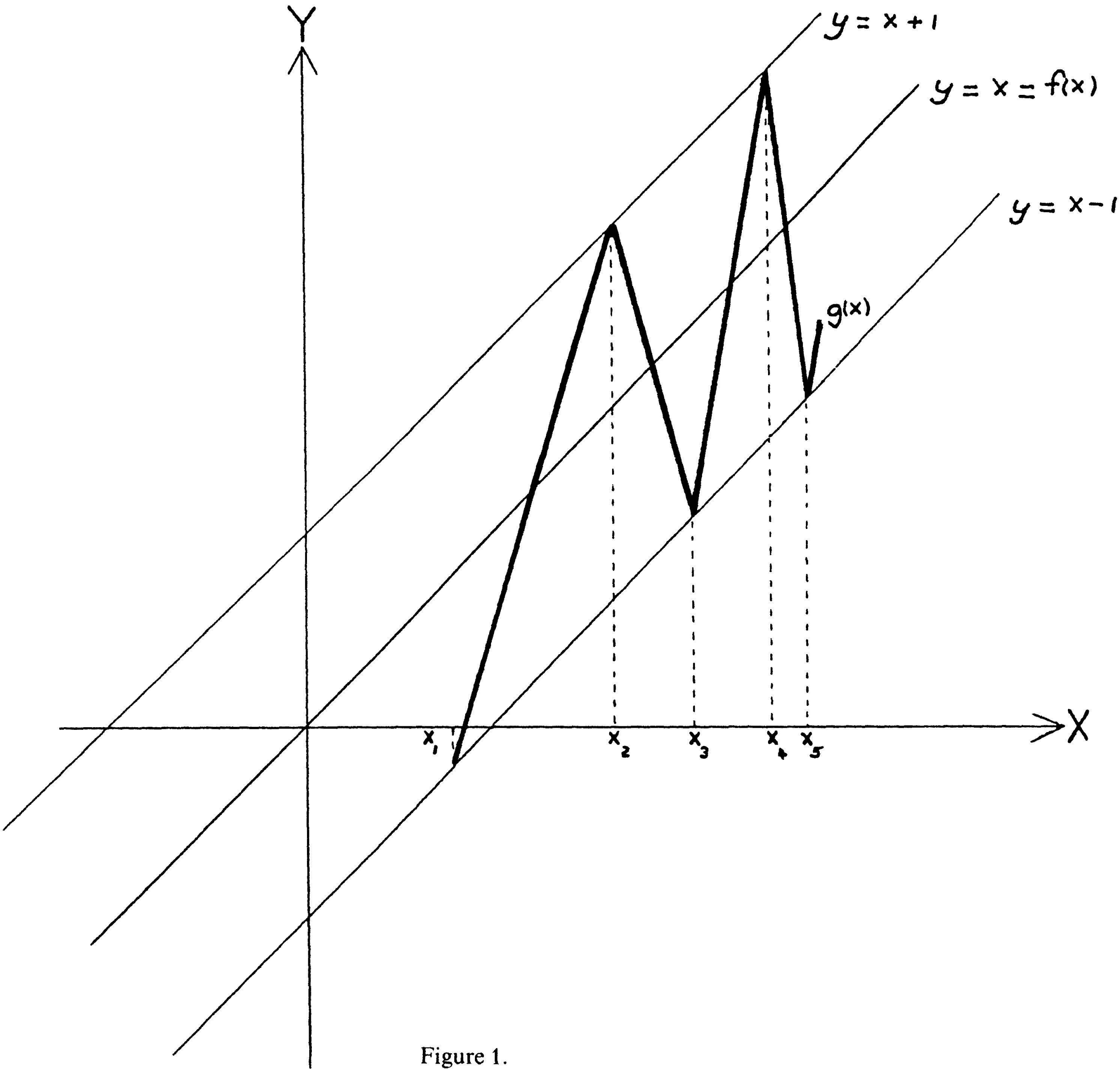

Figure 1.

and consequently, for $x > 1$,

$$\frac{x}{x+1} \leqq \frac{f(x)}{g(x)} \leqq \frac{x}{x-1}.$$

Hence $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = 1.$

4. Historical remarks and supplements

Although theorems like the ones discussed here bear the name of Marquis de L'Hôpital the rule was discovered by Johann Bernoulli. For sake of brevity let us call theorems which deduce monotonicity of a function from the sign of its derivative (or derivates) monotonicity theorems. The use of these in the proof of l'Hôpital's rule was made by Lettenmeyer [4]. Since monotonicity theorems are known to hold for Dini derivates, it is clear from our exposition that the right-hand derivatives can be replaced in Theorem 1–2 without affecting their validity by Dini derivates. The following counterexample:

$$f(x) = x + \sin x \cos x, \quad g(x) = f(x) e^{\sin x}$$

$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = 0$ and no limit for $\frac{f(x)}{g(x)}$ as $x \to \infty$ was given already in 1879 by O. Stolz [6], who also showed that Theorem 3 (with ordinary rather than one-sided derivatives) can be deduced from Theorem 2. A simple proof based on the Newton-Leibniz formula was given by Boas [2] but one may conjecture that the method was already known to Huntington [3].

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An extension of the isoperimetric inequality on the sphere

We shall consider the n-dimensional sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, endowed with the spherical distance function $d(x, y)$ and the (normalized) Lebesgue measure μ. For $x \in S^n$ and $0 \leq \theta \leq \pi$, the *spherical cap* of centre x and radius θ is $C(x, \theta) = \{y \in S^n : d(x, y) \leq \theta\}$. It is well known that if $A \subset S^n$ and $\mu(A) = \mu(C)$ for some spherical cap C, then the diameter of A is at least as large as the diameter of C. This is usually considered to be a variant of the isoperimetric inequality on the sphere S^n; it is, in fact, an immediate consequence of the isoperimetric inequality. Our aim is to extend this inequality and thereby answer a question raised by Paul Erdös [4].