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Autor: Felgner, Ulrich
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Estimates for the sequence of primes

Abstract. By elementary methods we obtain estimates for $\pi(x)$ and p_n which are considerably sharper than those previously obtained by similar methods.

1. Introduction. Let p_n be the n^{th} prime (thus $p_1 = 2, p_2 = 3, \ldots$) and let $\pi(x)$ be the number of primes p such that $p \leq x$. Various fairly precise estimates for p_n and $\pi(x)$ are known, notably those of J. B. Rosser and L. Schoenfeld [5]. However these estimates are proved by rather deep methods (location of the zeros of the Riemann zeta function) as well as by heavy computations. It is therefore desirable to obtain sufficiently good estimates by elementary methods. Such estimates are known, but it appears that it is possible to improve them. For example, a result of Rosser-Schoenfeld [5] states that

$$n \cdot \log(n) < p_n < n \cdot \log(n) + n \cdot \log(\log(n)) \quad \text{for} \quad 6 \leq n,$$

while W. Sierpiński [6] is satisfied with

$$0.25\, n \cdot \log(n) < p_n < 36\, n \cdot \log(n)$$

and Apostol [1] with $(1/6)\, n \cdot \log(n) < p_n < 6\, n \cdot \log(n)$. We shall prove here by similar elementary methods that for all $n \geq 3$:

$$0.91\, n \cdot \log(n) < p_n < 1.7\, n \cdot \log(n)\,.$$

To my knowledge all elementary estimates are based on arithmetical properties of the binomial coefficient $\binom{2n}{n}$. In our proofs we shall use certain multinomial coefficients instead. Thereby we obtain sharper bounds and also – surprisingly – simpler proofs.

Notation. log denotes natural logarithm to the base $e = 2.718\ldots$, and $[x]$ is the greatest integer m such that $m \leq x$. Furthermore, p and q are variables for primes, $m, n, k, \ldots$ are variables for positive integers and $x, y, z, \ldots$ are variables for reals.

2. Arithmetical properties of some binomial coefficients

Definition. For each real number $x \geq 1$ and each prime p let $\alpha_x(p) = \alpha(p)$ be the integer defined by $p^{\alpha(p)} \leq x < p^{1+\alpha(p)}$ and put

$$B(x) = \prod_{p \leq x} p^{\alpha(p)}\,.$$

Thus $B(x)$ is the least common multiple of the integers $1, 2, \ldots, [x]$. $B(x) = B([x])$ and $\alpha_n(p) = [\log_p(n)]$. The importance of $B(n)$ comes from the fact that

$n^{\pi(n)} = \prod_{p \leq n} n = \prod_{p \leq n} p^{\log_p(n)} \geq B(n) \geq \prod_{p \leq n} p$, where all terms are 'almost' the same.

Lemma 2.1. $n^2 \cdot \binom{6n}{3n} \cdot \binom{3n}{n} \leq B(1.2n) \cdot B(6n)$ for all $n \geq 1$.

Proof. Let $B(1.2n) = \prod p^{\alpha(p)}$, $B(6n) = \prod p^{\beta(p)}$, $n = \prod p^{\gamma(p)}$ and $\binom{6n}{3n} \cdot \binom{3n}{n} = \prod p^{\delta(p)}$ be the representations as products of prime powers. Put

$$D(n, p^j) = \left[\frac{6n}{p^j}\right] - \sum_{k=1}^{3} \left[\frac{kn}{p^j}\right].$$

By Legendre's Lemma (cf. Trost [7], p. 8–9)

$$\delta(p) = \sum_{j=1}^{\gamma(p)} D(n, p^j) + \sum_{j=\gamma(p)+1}^{\alpha(p)} D(n, p^j) + \sum_{j=\alpha(p)+1}^{\beta(p)} D(n, p^j),$$

It follows from

$$\frac{kn}{p^j} - 1 < \left[\frac{kn}{p^j}\right] \leq \frac{kn}{p^j}$$

that $D(n, p^j) \leq 2$. However, if $\alpha(p) + 1 \leq j \leq \beta(p)$, then $1.2n < p^j \leq 6n$ and hence $D(n, p^j) = 1$. If the prime p divides n then $D(n, p^j) = 0$ for all $j \leq \gamma(p)$. Thus $\delta(p) \leq 0 + 2(\alpha(p) - \gamma(p)) + (\beta(p) - \alpha(p)) = \alpha(p) + \beta(p) - 2\gamma(p)$ provided $p \mid n$, and $\delta(p) \leq \alpha(p) + \beta(p)$ otherwise. □

Lemma 2.2. $B(1806n) \leq B(n) \cdot \binom{1806n}{903n} \cdot \binom{903n}{301n} \cdot \binom{301n}{43n} \cdot \binom{43n}{n}$.

Proof. Put $B(n) = \prod p^{\alpha(p)}$, $B(1806n) = \prod p^{\beta(p)}$ and

$$\binom{1806n}{903n} \cdot \binom{903n}{301n} \cdot \binom{301n}{43n} \cdot \binom{43n}{n} = \binom{1806n}{903n, 602n, 258n, 42n, n} = \prod p^{\gamma(p)}$$

and

$$E(n, p^j) = \left[\frac{1806n}{p^j}\right] - \left(\left[\frac{n}{p^j}\right] + \left[\frac{42n}{p^j}\right] + \left[\frac{258n}{p^j}\right] + \left[\frac{602n}{p^j}\right] + \left[\frac{903n}{p^j}\right]\right).$$

By Legendre's Lemma $\gamma(p) = \sum E(n, p^j)$, where j runs from 1 to $\beta(p)$. We have to prove that $\beta(p) - \alpha(p) \leq \gamma(p)$ for all primes p.
If $1 \leq \beta(p) - \alpha(p)$ and $\alpha(p) < j \leq \beta(p)$, then $n < p^j \leq 1806n$ by the definition of $B(x)$. Select $a \in \mathbb{N}$ such that $1806n/(a+1) < p^j \leq 1806n/a$. Then

$$(*) \quad \left[\frac{1806n}{p^j}\right] = a, \quad \left[\frac{903n}{p^j}\right] = \left[\frac{a}{2}\right], \quad \left[\frac{602n}{p^j}\right] = \left[\frac{a}{3}\right], \quad \left[\frac{258n}{p^j}\right] = \left[\frac{a}{7}\right]$$

and $[42\,n/p^j] = [a/43]$. Therefore

$$\left[\frac{1806\,n}{p^j}\right] = a > \left[\frac{1805\,a}{1806}\right] = \left[\frac{42\,a + 258\,a + 602\,a + 903\,a}{1806}\right] \geq$$

$$\geq \left[\frac{42\,a}{1806}\right] + \left[\frac{a}{7}\right] + \left[\frac{a}{3}\right] + \left[\frac{a}{2}\right].$$

This together with (*) implies that $E(n, p^j) \geq 1$ whenever $n < p^j \leq 1806\,n$. Thus $\gamma(p) = \sum E(n, p^j) \geq (\beta(p) - \alpha(p)) \cdot 1$. $\square$

Lemma 2.3. For $1 \leq n$: $\dfrac{1}{3\sqrt{n}} \cdot \left(\dfrac{27}{4}\right)^n < \dbinom{3\,n}{n} < \dfrac{1}{2} \cdot \left(\dfrac{27}{4}\right)^n$.

Proof. Since our claim is true for $n = 1$ let us assume for induction that it is also true for some $n \geq 1$. Then

$$\binom{3\,(n+1)}{n+1} = \binom{3\,n}{n} \frac{3\,(9\,n^2 + 9\,n + 2)}{4\,n^2 + 6\,n + 2} > \frac{1}{\sqrt{n}} \left(\frac{27}{4}\right)^n \frac{9\,n^2 + 9\,n + 2}{4\,n^2 + 6\,n + 2} =$$

$$= \frac{1}{3\sqrt{n}} \left(\frac{27}{4}\right)^{n+1} \cdot \frac{36\,n^2 + 36\,n + 8}{36\,n^2 + 54\,n + 18} > \frac{1}{3\sqrt{n+1}} \cdot \left(\frac{27}{4}\right)^{n+1}.$$

The upper bound is confirmed similarly. $\square$

3. Estimates for the product of the primes

Theorem 3.1. For all real numbers $x \geq 1$: $B(x) < 3^x$.

Proof. We shall prove by induction that for all integers $n \geq 1$, $B(n) < 3^n$ holds (then the claim follows for all reals $x \geq 1$ since then $B(x) = B([x]) < 3^{[x]} \leq 3^x$).
With the aid of a personal computer it is not difficult to verify the claim for all positive integers $n \leq 126420 = 70 \cdot 1806$. In fact:

$$B(115089) \leq \ldots \leq B(126420) < 3^{115089} < \ldots < 3^{126420},$$
$$B(104323) \leq \ldots \leq B(115088) < 3^{104323} < \ldots < 3^{115088},$$
$$B(94940) \leq \ldots \leq B(104322) < 3^{94940} < \ldots < 3^{104322}, \text{ etc.}$$

Now, let $n = z \cdot 1806$, where $z \geq 70$, and assume for induction that $B(m) < 3^m$ for all $m \leq n$. By Lemma 2.2:

$$(*) \quad B(n+1) \leq B(n+2) \leq \ldots \leq B(n+1806) = B(1806 \cdot (z+1)) \leq B(z+1) \cdot M,$$

where M is a product of four binomial coefficients (in fact a multinomial coefficient) as in lemma 2.2. An upper bound for M is obtained as follows. Since generally we have

$$\binom{a(b+1)}{b+1} = \binom{a\,b}{b} \cdot \frac{a\,b+1}{(a-1)\,b+1} \cdot \frac{a\,b+2}{(a-1)\,b+2} \cdot \ldots \cdot \frac{a\,b+a-1}{(a-1)\,b+a-1} \cdot \frac{a(b+1)}{b+1}$$
$$\leq \binom{a\,b}{b} \cdot \frac{a^a}{(a-1)^{a-1}},$$

it follows inductively:

$$\binom{a\,b}{b} < a \cdot \left(\frac{a^a}{(a-1)^{a-1}}\right)^{b-1}.$$

Therefore in our case:

$$\binom{1806\,k}{903\,k} \cdot \binom{903\,k}{301\,k} \cdot \binom{301\,k}{43\,k} \cdot \binom{43\,k}{k} < 4^{452k} \cdot 3^{603k} \cdot 7^{259k} \cdot 43^{43k} \leq e^{1954.790\,596\,k}.$$

Together with the induction hypothesis $B(z+1) < 3^{z+1}$ we obtain hence from (*):

$$B(n+1) \leq B(n+2) \leq \ldots \leq B(n+1806) \leq 3^{z+1} \cdot e^{1954.790\,596\,(z+1)}.$$

But $e^{1954.790\,596\,(z+1)} < 3^{1805z}$ for $z \geq 70$ (take logarithms!). Hence

$$B(n+1) \leq \ldots \leq B(n+1806) = B(1806\,(z+1)) < 3^{1806z+1} = 3^{n+1} < \ldots .$$

This proves the theorem. $\square$

Theorem 3.2. For all real numbers $x \geq 13$: $2.2^x < B(x)$.

Proof. We shall prove that for all integers $n \geq 13$, $2.2^{n+1} < B(n)$ holds. Then the claim follows for all reals $x \geq 13$, since

$$2.2^x < 2.2^{[x]+1} < B([x]) = B(x).$$

With the aid of a personal computer it is easily seen that $2.2^{n+1} < B(n)$ is true for all integers n such that $13 \leq n \leq 1475$. For integers n such that $1476 = 6 \cdot 246 \leq n$ we proceed as follows. Choose $k \in \mathbb{N}$ such that $6\,k \leq n < 6\,(k+1)$, thus $k \geq 246$. By lemma 2.1, lemma 2.3 and the well-known fact $\binom{2m}{m} > 4^m/2\sqrt{m}$ (cf. Trost [7], p. 58) we have

$$B(1.2\,k) \cdot B(6\,k) > k^2 \cdot \binom{6\,k}{3\,k} \cdot \binom{3\,k}{k} \geq \frac{k^2 \cdot 4^{3k} \cdot 27^k}{2\sqrt{3\,k} \cdot 3\sqrt{k} \cdot 4^k} > 4^{2k} \cdot 3^{3k}$$

since $k > 11 > 6\sqrt{3}$. By theorem 3.1: $B(1{,}2k) < 3^{1.2k}$, hence $B(6k) > 4^{2k} \cdot 3^{1.8k} > e^{4.75k}$. But $e^{4.75k} > 2.2^{6(k+1)}$ for $k \geq 246$ (take logarithms!). Since $6k \leq n < 6(n+1)$ we conclude that

$$B(n) \geq B(6k) > e^{4.75k} > 2.2^{6(k+1)} \geq 2.2^{n+1}. \quad \square$$

If we restrict theorem 3.2 to integers we could say that $2.2^n < B(n)$ is true for all integers $n \geq 11$. A simple proof that $2^n \leq B(n) \leq 4^n$ for all integers $n \geq 7$ is given in M. Nair [4].

Theorem 3.3. For all integers $n \geq 1$: $\prod_{p \leq n} p < 3^n$, and for all integers $n \geq 41$: $2.1^n < \prod_{p \leq n} p$.

Proof. By theorem 3.1: $\prod_{p \leq n} p \leq B(n) < 3^n$ for all $1 \leq n \in \mathbb{N}$. In order to prove the lower bound notice first that for any prime number q, for any positive integers i,j and for any real numbers x, y the inequalities $q^i \leq x < q^{i+1}$ and $q^j \leq y < q^{j+1}$ imply $xy < q^{i+j+2}$. Hence $B(xy) \leq B(x) \cdot B(y) \cdot \prod_{q \leq xy} q$. Therefore if $11 \leq n \in \mathbb{N}$ then by theorems 3.1 and 3.2:

$$2.2^n < B(n) \leq B(\sqrt{n}) \cdot B(\sqrt{n}) \cdot \prod_{p \leq n} p < (3^{\sqrt{n}})^2 \cdot \prod_{p \leq n} p.$$

However $2.1^n \cdot 9^{\sqrt{n}} \leq 2.2^n$ for $n \geq 2231$ (take logarithms!) and we conclude that $2.1^n < \prod_{p \leq n} p$ for all $2231 \leq n \in \mathbb{N}$. With the aid of a personal computer it is easily seen that $2.1^n < \prod_{p \leq n} p$ also holds for all integers n such that $41 \leq n < 2231$. $\square$

We could strengthen theorem 3.2 to: $2.206^n < B(n)$ for all $n \geq 13$ with the same proof but a lot more computation. Then theorem 3.3 can be strengthened to $2.2^n < \prod_{p \leq n} p$ for all integers $n \geq 59$. Again the same proof works but computations have to be carried out for all n such that $59 \leq n \leq 650841$. Rosser and Schoenfeld [5] proved that $2.316^n < \prod_{p \leq n} p < 2.763^n$ for $n \geq 101$ in the case of the lower bound.

4. Estimates for p_n and $\pi(x)$

We shall begin with a simple proof of (an extension of) Bertrand's postulate.

Theorem 4.1. For all integers $n \geq 8$ there is a prime p such that $n < p \leq 1.5n$.

Proof. For reals $a < b$ let $A(a; b)$ be the product of all primes p such that $a < p \leq b$. Then by theorem 3.3: for $n \geq 28$:

$$A(n; 1.5n) = A(1; 1.5n)/A(1; n) \geq 2.1^{1.5n}/3^n = (3.043\ldots/3)^n > 1.$$

The product $A(n; 1.5n)$ is hence non-empty which means that there is a prime between n and $3n/2$. The claim is obviously also true for all n such that $8 \leq n \leq 28$. $\square$

It follows from theorem 4.1 that $p_{m+1} \leq 1.5\, p_m$ for $p_m \geq p_5 = 11$. Hence

$$p_{n+2} \leq \tfrac{1}{2}(p_{n+1} + 2p_{n+1}) \leq \tfrac{1}{2}(\tfrac{3}{2}p_n + 2p_{n+1}) \leq p_n + p_{n+1}$$

for $n \geq 5$ (even for $n \geq 2$). This is Ishikawa's theorem (cf. Trost [7], Satz 35).
Theorem 4.1 does not tell us how many primes there are between n and $\frac{3}{2}n$. For the number of primes between n and $2n$ however we obtain the following estimate:

$$\forall x \geq 6\colon \frac{3}{5} \cdot \frac{x}{\log(2x)} < \pi(2x) - \pi(x)\,.$$

To prove this we argue as in Trost [7], Satz 31: let P_n denote the product of all primes p such that $n < p \leq 2n$ and put $P_n \cdot Q_n = \binom{2n}{n}$. By theorem 3.3: $Q_n \leq 3^{2n/3} \cdot (2n)^{\sqrt{n/2}}$, hence

$$P_n > \frac{e^{0.653\,886\,169\,n}}{\sqrt{4n}} \cdot \left(\frac{1}{2n}\right)^{\sqrt{n/2}} = (2n)^x$$

Taking logarithms we see that $x \cdot \log(2n) > 0{,}6\,n$ for $n \geq 19441$, from which our claim follows for all $n \geq 19441$. A simple check of tables shows that our claim is true even for all $x \geq 6$.
Let us mention that Hua Loo Keng [3], p. 85, proves that $0.023\, n/\log(2n) < \pi(2n) - \pi(n)$ and Finsler $n/3\log(2n) < \pi(2n) - \pi(n)$ (cf. Trost [7], Satz 32). Rosser and Schoenfeld [5] state that $(3/5) \cdot x/\log x < \pi(2x) - \pi(x)$.

Theorem 4.2.

(i) $0.788 \dfrac{x}{\log(x)} < \pi(x)$ for all reals $x \geq 5$.

(ii) $\pi(x) < 1.5 \dfrac{x}{\log(x)}$ for all reals $x \geq 2$.

Proof. (i) Since $2.2^x < B(x) = B([x]) \leq [x]^{\pi([x])} \leq x^{\pi(x)}$ by theorem 3.2 for $x \geq 13$ the claim follows by taking logarithms and a direct check in the case $5 \leq x < 13$.
(ii) A simple check of tables shows that our claim is true for all integers n such that $2 \leq n \leq 103854 = 6 \cdot 17309$. In fact:

$$\pi(2) \leq \ldots \leq \pi(10) = 4 < 1.5 \cdot 2/\log 2 < \ldots < 1.5 \cdot 10/\log 10\,,$$

etc. (notice that $\pi(103854) = 9919$). Now let $k \geq 17309$ and assume for induction that $\pi(n) < 3n/2\log(n)$ is true for $2 \leq n \leq 6k$.
If q is a prime such that $k < q \leq 6k$ then q divides $\binom{6k}{3k}\binom{3k}{k}$ (see lemma 5.2). Therefore by lemma 2.3:

$$k^{\pi(6k)-\pi(k)} \leq \prod_{k<p\leq 6k} p \leq \binom{6k}{3k}\binom{3k}{k} < 4^{3k} \cdot \frac{1}{2} \cdot \left(\frac{27}{4}\right)^k < e^{6.068\,425\,589\,k}\,.$$

Taking logarithms and using the induction hypothesis for $\pi(k)$ we get

$$\pi(6k) < \frac{7.568\,425\,589\,k}{\log k} = 1.261\,404\,266\,\frac{6k}{\log k} < 1.493\,\frac{6k}{\log(6k)}$$

where the last inequality holds since $k \geq 17\,309$. We conclude that

$$\begin{aligned}\pi(6k+6) = \pi(6k+5) \leq \pi(6k+4) + 1 = \pi(6k+3) + 1 = \pi(6k+2) + 1 = \\ = \pi(6k+1) + 1 \leq \pi(6k) + 2 \leq 2 + 1.493\,\frac{6k}{\log(6k)} < \\ < 1.5\,\frac{6k+1}{\log(6k+1)} < \ldots < 1.5\,\frac{6k+6}{\log(6k+6)}\end{aligned}$$

(again for $k \geq 17\,309$). □
Let us mention that Apostol [1] proves $n/6\log n < \pi(n) < 6n/\log n$ and Trost [7] $2n/3\log n < \pi(n) < 1.6\,n/\log n$ based on checking of tables for $n \leq 70\,000$. The following lemma is due to N. Costa Pereira [2].

Lemma 4.3. For $n \geq 1$: $n^n \leq B(p_n)$.

Proof. $m \cdot \log(p_m) = \pi(p_m) \cdot \log(p_m) < 1.5\,p_m$ by theorem 4.2 (ii). Since $1.5\,e < \log(p_m)$ for $p_m \geq 59$ we obtain $e\,m < p_m$ for $p_m \geq 59$. In fact $e\,m < p_m$ holds for all $m \geq 10$ (i.e. $p_m \geq 29$).
For $1 \leq n < 10$ it is easy to verify that $n^n \leq B(p_n)$. Now let $n \geq 10$ be given and asumme for induction that the claim of the lemma is true for all $k < n$. Then $B(p_n) \geq p_n \cdot B(p_{n-1})$ $\geq e\,n \cdot (n-1)^{n-1}$. But $e > \left(\frac{n}{n-1}\right)^{n-1}$, hence $B(p_n) \geq n^n$. □

Theorem 4.4. For $n \geq 3$: $0.91\,n \cdot \log n < p_n < 1.7\,n \cdot \log n$.

Proof. $n^n \leq B(p_n) < 3^{p_n}$ by theorem 3.1 and lemma 4.3. By taking logarithms it follows that $0.91\,n \cdot \log n < (n \cdot \log n)/\log 3 < p_n$.
Concerning the upper bound of p_n notice first that $1.7\log n \geq 13$ for $n \geq 2095$ and $(\log z)/z \leq 0.199\,764\,706$ for $z \geq 13$. Hence $(\log(1.7\log n))/(1.7\log n) \leq 0.199\,764\,706$ for $n \geq 2095$. By theorem 4.2 (i): $1/\pi(x) < (\log x)/(0.788\,x)$ for $x \geq 5$. For $x = 1.7\,n \cdot \log n$ we obtain therefore

$$\begin{aligned}\frac{n}{\pi(1.7\,n\log n)} &< \frac{n \cdot \log(1.7\,n\log n)}{0.788 \cdot 1.7 \cdot n \cdot \log n} = \\ &= \frac{n \cdot \log n}{0.788 \cdot 1.7 \cdot n \cdot \log n} + \frac{n \cdot \log(1.7 \cdot \log n)}{0.788 \cdot 1.7\,n \cdot \log n} \leq 1\end{aligned}$$

for $n \geq 2095$, *that is* $n < \pi(1.7\,n\log n)$. Since

$$p_n \leq 1.7\,n\log n \Leftrightarrow n = \pi(p_n) \leq \pi(1.7\,n\log n)$$

the claim of the theorem is proved for all $n \geq 2095$. An easy check of tables shows that our claim is even true for all $n \geq 3$. In fact $p_{1474} = 12343 < \ldots < p_{2094} = 18269 < 1.7 \cdot 1474 \cdot \log(1474) < \ldots$, etc. $\square$

The argument of theorem 4.4 shows that $p_n < 1.5\,n \cdot \log(n)$ for all $n \geq 127888158$ (replace 1.7 by 1.5 in the proof). More generally the argument shows that for any constant $k > 1/0.788 = 1.26903553\ldots$ there exists an integer M such that $p_n < k \cdot n \cdot \log n$ for all $n \geq M$.

5. Final remarks

The source for our estimates of p_n and $\pi(x)$ are lemma 2.1 and lemma 2.2. Both lemmata might seem a bit obscure and we feel that we should motivate the truth of them.

All that is needed for a full understanding of these lemmata is a precise knowledge of the prime divisors of the binomial coefficients. Let p be a prime. If $n < p \leq 2n$ then p divides $\binom{2n}{n}$. Erdös observed that p does not divide $\binom{2n}{n}$ if $\frac{2}{3}n < p \leq n$. The following lemma is an extension of this observation.

Lemma 5.1. Let p be a prime and $1 \leq k \in \mathbb{N}$, $k < n$.

(i) If $k \leq p$ and $n/k < p \leq 2n/(2k-1)$ then p divides $\binom{2n}{n}$.

(ii) If $2k-1 \leq p$, $k \geq 2$ and $2n/(2k-1) < p \leq n/(k-1)$, then p does not divide $\binom{2n}{n}$.

Proof. $1 \leq k < n$ implies $n/k < 2n/(2k-1) < n/(k-1)$. Ad (i): by assumption $n < kp$, hence $p^k \nmid n!$ (since $p \nmid k-1$) and $(k-1)\,p < n$, hence $p^{k-1} \mid n!$. Again by assumption $p^{2k-1} \mid (2n)!$ and the claim follows. Ad (ii): Similarly $p^{k-1} \mid n!$, $p^k \nmid n!$, $p^{2k-1} \nmid (2n)!$, $p^{2k-2} \mid (2n)!$. $\square$

Lemma 5.2. Let p be a prime and $2 \leq k < n$.

(i) If $k < p$ and $2n/(k+1) < p \leq 3n/(1 + [3k/2])$ then p divides $\binom{3n}{n}$.

(ii) If $3n/(1 + [3k/2]) < p \leq 2n/k$ then p does not divide $\binom{3n}{n}$.

Proof. Notice first that

$$\frac{2n}{k+2} < \frac{3n}{1+[3(k+1)/2]} < \frac{2n}{k+1} < \frac{3n}{1+[3k/2]} < \frac{2n}{k} < \frac{3n}{1+[3(k-1)/2]} < \frac{2n}{k-1}.$$

From this both claims follow as in lemma 5.1. $\square$

It follows from 5.1 and 5.2 that the product of all primes p in the interval $n < p \leq 6n$ divides $\binom{6n}{3n} \cdot \binom{3n}{n}$. If p^2 divides $\binom{6n}{3n}\binom{3n}{n}$ then $p \leq 1.2\,n$. Thus products of primes are much better estimated by the multinomial coefficient $\binom{6n}{3n}\cdot\binom{3n}{n}=\binom{6n}{n,2n,3n}$ than by $\binom{2n}{n}$ as it is usually done. An even better approximation is furnished by the multinomial coefficient used in lemma 2.2.

In conclusion I would like to offer to those who like primes with nice digit patterns two new examples, namely $p = 122333221$ and $q = 1223334444555554444333221$.

Ulrich Felgner, Mathematisches Institut der Universität Tübingen

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Kleine Mitteilungen

Zu K. Schüttes Verallgemeinerung des Satzes von Napoleon

In [2] und [3] bewies K. Schütte den folgenden

Satz 1 *Den Seiten des Dreiecks $A_1A_2A_3$ seien Dreiecke $A_2A_3B_1$, $A_3A_1B_2$, $A_1A_2B_3$ aufgesetzt, und zwar entweder alle nach außen oder alle nach innen. Für die Innenwinkel β_i bei B_i, $i = 1, 2, 3$ gelte $\beta_1 + \beta_2 + \beta_3 = \pi$ (siehe Abb. 1). Dann bilden die Umkreismittelpunkte M_1, M_2, M_3 der Aufsatzdreiecke ein Dreieck mit den Innenwinkeln β_1, β_2 und β_3, sofern nicht alle drei Umkreismitten zusammenfallen.*

Im folgenden wird ein vereinfachter Beweis dieses Satzes gezeigt:

Nach dem Satz vom Zentriwinkel ist die Umkreismitte M_i für jede gerade Permutation (i,j,k) von $(1,2,3)$ das Zentrum einer Drehung δ_i durch den Winkel $2\beta_i$, die A_j in A_k überführt. Wegen der vorausgesetzten Lage der Aufsatzdreiecke erfolgen alle drei Drehungen in demselben Sinn. Das Produkt $\delta_1\delta_2\delta_3$ ist eine Bewegung mit dem Drehwinkel 2π, die A_2 fix läßt, also die Identität. Da – wie vorausgesetzt – die Drehzentren verschieden sind, bilden sie ein Dreieck $M_1M_2M_3$ mit den halben Drehwinkeln β_1, β_2 und β_3 als Innenwinkeln. Dies läßt sich bekanntlich wie folgt zeigen: