Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 46 (1991)
Heft: 2

Rubrik: Kleine Mitteilungen

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Siehe Rechtliche Hinweise.

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. Voir Informations légales.

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. See Legal notice.

Download PDF: 01.04.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Kleine Mitteilungen

On an integral inequality

1. Introduction

The following is an interesting elementary inequality which was posed as a problem of the 34th Putnam Competition 1973:
If f is differentiable on $[0,1]$ with $f(0)=0$ and $0<f'(x)\leq 1$ for all $x\in[0,1]$ then

$$\int_0^1 f^3(x)\,dx \leq \left(\int_0^1 f(x)\,dx\right)^2. \tag{1}$$

Furthermore, the following inequality appeared as Problem P338 in Canad. Math. Bull. *26* (June 1983):
Let $f:[0,1]\to \boldsymbol{R}$ be differentiable such that $f(0)=0$ and $0\leq f'(x)\leq 1$ whenever $x\in[0,1]$. If $p\geq 1$ then

$$\left(\int_0^1 f(x)\,dx\right)^p \geq p\,2^{1-p}\int_0^1 f(x)^{2p-1}\,dx. \tag{2}$$

For $0<p<1$ the reverse inequality is valid.
Of course for $p=2$, (2) becomes (1).
In this note we shall prove extensions of inequalities (1) and (2).

2. Results

i) Let $a>0$ and $w:[0,a]\to[0,\infty)$ be an integrable function and $f:[0,a]\to[0,\infty)$ be differentiable such that $f(0)=0$. We define

$$F(x):=\left(\int_0^x w(t)f(t)\,dt\right)^p - p\,2^{1-p}\int_0^x w(t)f(t)^{2p-1}\,dt. \tag{3}$$

Then $F(0)=0$ and

$$F'(x)=p\,w(x)f(x)\left[\left(\int_0^x w(t)f(t)\,dt\right)^{p-1} - 2^{1-p}f(x)^{2(p-1)}\right].$$

Let $p>1$ and put

$$G(x):=\int_0^x w(t)f(t)\,dt - f^2(x)/2.$$

Then $G(0) = 0$ and $G'(x) = f(x)[w(x) - f'(x)]$. If there holds

$$0 \leq f'(x) \leq w(x) \tag{4}$$

we get $G'(x) \geq 0$. Thus we conclude $G(x) \geq 0$, $F'(x) \geq 0$ and finally $F(x) \geq 0$. If

$$f'(x) \geq w(x) \tag{5}$$

we get $G'(x) \leq 0$ and as above $F(x) \leq 0$.
Analogously we can consider the cases $0 < p < 1$ or $p < 0$. Therefore the following theorem is valid.

Theorem 1

Let $f\colon [0, a] \to \boldsymbol{R}$ *be differentiable such that* $f(0) = 0$. *Then*

$$\left(\int_0^a w(x) f(x)\,dx\right)^p \geq p\,2^{1-p} \int_0^a w(x) f(x)^{2p-1}\,dx \tag{6}$$

if $0 \leq f'(x) \leq w(x)$ *and* $p > 1$ *or* $p < 0$ *or* $f'(x) \geq w(x)$ *and* $0 < p < 1$. *The reverse inequality is valid for* $0 \leq f'(x) \leq w(x)$ *and* $0 < p < 1$ *or* $f'(x) \geq w(x)$ *and* $p > 1$ or $p < 0$. ■

($a = 1$ and $w(x) \equiv 1$ yield inequality (2).)

ii) For another generalisation of (1) let $M > 0$ and

$$f(0) = 0 \quad \text{and} \quad 0 \leq f'(x) \leq M \quad \text{for all} \quad x \in [0, a]. \tag{7}$$

Then $0 \leq f(x) \leq M x$ and $0 \leq \int_0^x f(t)\,dt \leq M x^2/2$ for $0 \leq x \leq a$.
We now define (for suitable p and r)

$$F(x) := \left(\int_0^x f(t)\,dt\right)^p - \int_0^x f(t)^r\,dt.$$

Then $F(0) = 0$ and $F'(x) = f(x) g(x)$, where

$$g(x) = p\left(\int_0^x f(t)\,dt\right)^{p-1} - f(x)^{r-1}.$$

Clearly, $g(0) = 0$ and

$$g'(x) = f(x)\left[p(p-1)\left(\int_0^x f(t)\,dt\right)^{p-2} - (r-1) f(x)^{r-3} f'(x)\right].$$

Let $1 < p \leq 2$ and $r \geq 3$. Then

$$g'(x) \geq f(x)[p(p-1)(Mx^2/2)^{p-2} - (r-1)M^{r-2}x^{r-3}]$$
$$= f(x)x^{2p-4}M^{r-2}[p(p-1)2^{2-p}M^{p-r} - (r-1)x^{r-2p+1}].$$

Thus, if

$$0 < a \leq [p(p-1)2^{2-p}M^{p-r}/(r-1)]^{1/(r-2p+1)} \tag{8}$$

we have $g'(x) \geq 0$, $g(x) \geq 0$, $F'(x) \geq 0$ and finally $F(x) \geq 0$ Therefore we have proved the following

Theorem 2

Let $1 < p \leq 2$ *and* $r \geq 3$. *The differentiable function* f: $[0, a] \to \boldsymbol{R}$ *satisfies* $f(0) = 0$ *and* $0 \leq f'(x) \leq M$ *for all* $0 \leq x \leq a$, *a subject to* (8). *Then*

$$\left(\int_0^a f(x)\,dx\right)^p \geq \int_0^a f(x)^r\,dx. \tag{9}$$

If $f'(x) \geq M$ *the reverse inequality holds true.* ■

For $M = 1$ we have a result of P. R. Beesack (see [1]).

Remark. For another generalisation of (1) see [2] and [3].

Josip E. Pečarić, University of Zagreb
Walther Janous, Ursulinengymnasium Innsbruck

REFERENCES

[1] Mitrinović D. S. and Pečarić J. E.: Monotone funkcije i njihove nejednakosti. Beograd 1990.
[2] Klamkin M. S. and Newman D. J.: Inequalities and identities for sums and integrals. Amer. Math. Monthly *83*, 26–30 (1975).
[3] Pečarić J. E.: Note on an integral inequality of M. S. Klamkin and D. J. Newman. ANU BIH Radovi, Odj. Pr. Mat. Nauka *78*, 91–93 (1985).

© 1991 Birkhäuser Verlag, Basel 0013-6018/91/02040-07 $ 1.50 + 0.20/0