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LITERATUR

1 Bohne E.: Erzeugung isophotischer Streifen bei geometrischer Zentralbeleuchtung zweiter Art, wenn die Leitkurve eine Gerade ist. Beitr Algebra Geom. *13*, 55–60 (1982).
2 Bohne E.: Spezielle ebene Kurven als Leitlinien isophotischer Zylinderstreifen. Beitr. Algebra Geom. *15*, 91–95 (1983).
3 Bohne E., Geise, G.: Zur Begriffswelt der Beleuchtungsgeometrie. Beitr. Algebra Geom. *20*, 117–122 (1985).
4 Caratheodory C.: Variationsrechnung und partielle Differentialgleichungen 1. Ordnung, Band 1. Teubner, Leipzig 1956.
5 de Jans C.: Les multiplicatrices de Clairaut. Gent 1912.
6 Lie S., Scheffers G.: Geometrie der Berührungsformationen, Band 1. Teubner, Leipzig 1896.
7 Loria G.: Spezielle algebraische und transzendente ebene Kurven, Band 1. Teubner, Leipzig 1910.
8 Meyer E.: Beleuchtungstechnik. Vieweg, Braunschweig 1938.
9 Paukowitsch P.: Flächen konstanter Beleuchtungsstärke. Beitr. Algebra Geom. *26*, 203–216 (1988).

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# On permutations involving pairs of twins

We are concerned with permutations of a set of $2n$ elements coupled into $n$ pairs; the members of each pair are referred to as *twins* (two decks of cards serve as a model). Let $p(n, k)$ be the probability that, in a randomly chosen permutation of such a set, exactly $k$ pairs of twins are nonseparated (occupy neighbouring positions). To be more precise: assume $S = \{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\}$ is the set, element $a_i$ matching $b_i$; define $T(n, k)$ as the set of all maps $\varphi$ of $\{1, \ldots, 2n\}$ onto $S$ such that $|\varphi^{-1}(a_i) - \varphi^{-1}(b_i)| = 1$ holds for exactly $k$ values of $i$ (permutations regarded as enumerations). Then

$$p(n, k) = \frac{1}{(2n)!} |T(n, k)| \qquad (0 \le k \le n)$$

(here and in the sequel $|\cdot|$ denotes the cardinality of a set). Evidently,

$$\sum_{k=0}^{n} p(n, k) = 1 \quad \text{for} \quad n = 1, 2, 3, \ldots . \tag{1}$$

It is not hard to show (see below) that $p(n, 0) < 1/2$, for any $n$ (this was one of the problems at the 30-th IMO in Germany, 1989, proposed by the author).

In this note we examine the asymptotic behaviour of quantities $p(n, k)$ for growing $n$. Namely, we prove

**Proposition**

$$\lim_{n \to \infty} p(n, k) = \frac{1}{ek!} \quad \text{for} \quad k = 0, 1, 2, \ldots .$$

We first derive two recursion formulas (Lemmas 1 and 2); it is convenient to define

$$T(n,-1)=\emptyset, \quad p(n,-1)=0; \qquad T(0,0)=\text{(singleton)}, \quad p(0,0)=1\,. \tag{2}$$

**Lemma 1**

$$k\,p(n,k)=p(n,k-1)+\frac{p(n-1,k-1)-p(n-1,k-2)}{2n-1} \quad \text{for} \quad n\geq k\geq 1\,.$$

**Proof.** Let $T'(n,k)$ be the set of all $\varphi$ in $T(n,k)$ such that $\varphi(1)$ and $\varphi(2)$ are not twins (for instance, $(a_3,a_2,b_2,b_4,a_4,a_1,b_3,b_1)$ belongs in $T'(4,2)$ while $(b_3,a_3,a_1,b_4,b_2,a_2,a_4,b_1)$ does not). Fix $n$ and $k$ with $n\geq k\geq 1$. We define a map

$$f\colon\ T'(n,k-1)\ \to\ T(n,k)$$

as follows. Given $\varphi\in T'(n,k-1)$, let $\varphi(j)$ be the twin of $\varphi(1)$; $j>2$ by the definition of $T'(\cdot,\cdot)$. We set

$$(f\varphi)(i)=\begin{cases}\varphi(i+1) & \text{for} \quad i<j-1,\\ \varphi(1) & \text{for} \quad i=j-1,\\ \varphi(i) & \text{for} \quad i\geq j;\end{cases}$$

that means, the initial element is moved to the position immediately preceding its twin. The resulting permutation indeed belongs to $T(n,k)$: all twin pairs which were not separated by $\varphi$ remains so under $f\varphi$, and a new neighbouring pair $\varphi(1)$, $\varphi(j)$ appears.
Choose a permutation $\psi\in T(n,k)$. If $\psi\in T'(n,k)$, then $|f^{-1}(\{\psi\})|=k$, i.e., $\psi=f\varphi$ for $k$ distinct permutations $\varphi\in T'(n,k-1)$; these preimages are obtained by moving the left member of any one of the $k$ neighbouring twin pairs of $\psi$ to the first position.
If $\psi\in T(n,k)\setminus T'(n,k)$, then $|f^{-1}(\{\psi\})|=k-1$; the same argument as above applies to all neighbouring twin pairs in $\psi$ except the leftmost one. Consequently

$$|T'(n,k-1)|=\sum_{\psi\in T(n,k)}|f^{-1}(\{\psi\})|=k\,|T'(n,k)|+(k-1)\,|T(n,k)\setminus T'(n,k)|\,. \tag{3}$$

Note that every permutation in $T(n,k)\setminus T'(n,k)$ arises by adjoining a twin pair (on the two initial positions) to a permutation of type $T(n-1,k-1)$. The adjoined pair can be any one of the $n$ pairs $a_i$, $b_i$ and can be arranged in two ways. Therefore

$$|T(n,k)\setminus T'(n,k)|=2n\,|T(n-1,k-1)|$$

and so

$$|T'(n,k)|=|T(n,k)|-2n\,|T(n-1,k-1)|$$

(true also for $n=k=1$, as well as for $k=0$ and any $n$, in agreement with (2)). Inserting this and an analogous equality with $k-1$ in place of $k$ into (3) we get after simple calculation

$$k|T(n,k)| = |T(n,k-1)| + 2n(|T(n-1,k-1)| - |T(n-1,k-2)|)$$

for $k \geq 1$. Division by $(2n)!$ yields the claimed equality, ending the proof of the lemma.

**Corollary**

$$p(n,1) = p(n,0) + \frac{p(n-1,0)}{2n-1} \quad \text{for} \quad n \geq 1, \tag{4}$$

$$p(n-1,1) = p(n-1,0) + \frac{p(n-2,0)}{2n-3} \quad \text{for} \quad n \geq 2, \tag{5}$$

$$p(n,1) \geq p(n,0) \quad \text{for} \quad n \geq 1. \tag{6}$$

**Proof.** (4) is the formula of Lemma 1 for $k=1$. (5) is nothing else than (4) with $n$ replaced by $n-1$. (6) is immediate from (4).

Note that (6) implies $p(n,0) < 1/2$ (the IMO problem mentioned at the beginning).

**Lemma 2**

$$p(n,0) = p(n-1,0) + \frac{p(n-2,0)}{(2n-3)(2n-1)} \quad \text{for} \quad n \geq 2;$$

$$p(0,0) = 1, \qquad p(1,0) = 0.$$

**Proof.** $p(0,0)=1$ by agreement; $p(1,0)=0$ is obvious from the definition of $p(\cdot,\cdot)$.

Fix $n \geq 2$ and consider any permutation of type $T(n,1)$. By removing the unique neighbouring twin pair we obtain a permutation of type $T(n-1,0)$ or $T(n-1,1)$; the latter occurs in the case when another pair of twins were separated by the removed pair. Reversing the argument, we see that any permutation of type $T(n,1)$ arises either from a permutation of type $T(n-1,0)$ by inserting a new pair and placing it in any one of the $2n-1$ sockets, or from a permutation of type $T(n-1,1)$ by inserting a new pair so as to disconnect the single hitherto neighbouring twin pair. Since the new pair can be chosen out of $n$ possibilities and can be arrayed in two ways, we arrive at the equality

$$|T(n,1)| = 2n((2n-1)|T(n-1,0)| + |T(n-1,1)|).$$

Division by $(2n)!$ yields

$$p(n,1) = p(n-1,0) + \frac{p(n-1,1)}{2n-1}.$$

Equating the right sides of the last equality and (4) we get

$$p(n,0)-p(n-1,0)=\frac{p(n-1,1)-p(n-1,0)}{2n-1}. \tag{7}$$

This, in view of (5), is precisely the formula of the lemma.

**Corollary**

The limits $p_k = \lim_{n\to\infty} p(n,k)$ exist and satisfy

$$p_k=\frac{1}{k!}p_0 \quad \text{for} \quad k=0,1,2,\ldots. \tag{8}$$

**Proof.** Lemma 2 shows that $p(n,0)$ increase with $n$. Therefore $p_0$ exists. Letting $n\to\infty$ in the formula of Lemma 1 we see that the existence of $p_{k-1}$ implies that of $p_k$, together with the equality $kp_k=p_{k-1}$. Hence, (8) results by induction.

Now, the recursion formula of Lemma 2 defines the 0-th column of the triangular array

$$\begin{array}{llllll}
p(0,0)= & 1 & & & & \\
 & 0 & 1 & & & \\
 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & \\
 & \frac{5}{15} & \frac{6}{15} & \frac{3}{15} & \frac{1}{15} & \\
 & \frac{36}{105} & \frac{41}{105} & \frac{21}{105} & \frac{6}{105} & \frac{1}{105} \\
 & \ldots & \ldots & \ldots & \ldots & \ldots \\
 & p(n,0) & \ldots & p(n,k) & \ldots & p(n,n) \\
 & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

The remaining entries are determined from the formula of Lemma 1.
All row sums are equal 1. Columns tend to limits $p_0, p_1, p_2, \ldots$. If we could assert that also

$$\sum_{k=0}^{\infty} p_k = 1 \tag{9}$$

we would be done, knowing (8). Passing to the limit, however, is not automatic; the convergence is not monotone, and it is not at all clear that the rows are summably-dominated. Thus, all that remains is to justify equality (9).

**Proof of the Proposition.** We begin by showing that

$$p(n,1)\geq p(n,2) \quad \text{for} \quad n\geq 2. \tag{10}$$

Using successively Lemma 1, then formula (7) (which is an equivalent form of the statement of Lemma 2), and finally estimate (6), we obtain

$$2p(n,2) = p(n,1) + \frac{p(n-1,1)-p(n-1,0)}{2n-1}$$

$$= p(n,1)+p(n,0)-p(n-1,0) \le p(n,1)+p(n,0) \le 2p(n,1),$$

which settles (10).
Now we prove

$$kp(n,k) \le p(n,k-1) \quad \text{for} \quad n \ge k \ge 3. \tag{11}$$

By Lemma 1, (11) is equivalent to

$$p(n-1,k-1) \le p(n-1,k-2) \quad \text{for} \quad n \ge k \ge 3. \tag{12}$$

For $k=3$, the inequality in (12) is implied by (10). Assume (12) for a certain $k \ge 3$ (and all $n \ge k$). For this $k$ (and all $n \ge k$) we then have inequality (11) satisfied, which clearly implies $p(n,k) \le p(n,k-1)$; and this is nothing else than (12) with $k-1$ replaced by $k$. Thus the inequality in (12) results by induction for all $k \ge 3$ (and all $n \ge k$). Hence, (11) is proved. From (11) we obtain by obvious induction

$$p(n,k) \le \frac{2}{k!}\, p(n,2) \quad \text{for} \quad n \ge k \ge 2.$$

Fix $m \ge 2$. We get for any $n > m$

$$\sum_{k=m+1}^{n} p(n,k) \le 2p(n,2) \sum_{k=m+1}^{n} \frac{1}{k!} < \frac{2p(n,2)}{m \cdot m!} \le \frac{1}{m!}.$$

Hence, by (1),

$$\sum_{k=0}^{m} p(n,k) > 1 - \frac{1}{m!}.$$

Letting $n \to \infty$ and then $m \to \infty$ we obtain

$$\sum_{k=0}^{\infty} p_k \ge 1.$$

The opposite inequality ($\sum p_k \le 1$) also holds; it follows from (1) e.g. by Fatou's lemma. The desired equality (9) is established.

The final conclusion

$$p_k = \frac{1}{ek!} \quad \text{for} \quad k=0, 1, 2, \ldots.$$

is a direct consequence of the relations (8) and (9).

Marcin E. Kuczma, Institute of Mathematics, University of Warsaw

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# Kleine Mitteilungen

## Eine Bemerkung über Iterationsverfahren

In dieser Note soll an Beispielen gezeigt werden, dass eine in der Literatur der numerischen Mathematik (etwa in [1], [2]) oft vorgebrachte Idee zur Konvergenzbeschleunigung für Iterationsverfahren sogar zu deren Divergenz führen kann.
Dort wird vorgeschlagen für das Iterationsverfahren

$$x_{i+1} = \Phi(x_i) \tag{1}$$

mit $x = (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in \mathbb{R}^n, \quad \Phi\colon \mathbb{R}^n \to \mathbb{R}^n$

die Konvergenz zu verbessern, indem dieses in

$$x_{i+1}^{(m)} = \Phi^{(m)}(x_{i+1}^{(1)}, \ldots, x_{i+1}^{(m-1)}, x_i^{(m)}, \ldots, x_i^{(n)}) \qquad m = 1, 2, \ldots, n \tag{2}$$

abgeändert wird. Die verbesserten Werte sollen also komponentenweise sofort zur weiteren Rechnung verwendet werden. Auch eine andere Komponentenreihenfolge im Sinne einer optimalen Auswahlstrategie mit Blick auf eine bessere Konvergenz sei denkbar. *Die Variante* (2) *kann sogar im konvergenten Fall des gewöhnlichen Verfahrens* (1) *zur Divergenz führen.* Die vorgeschlagene Methode (2) erfordert nebst den üblichen Konvergenzbedingungen von (1) (Kontraktionseigenschaften im linearen Fall) [3] von Fall zu Fall gesonderte Untersuchungen. Dies demonstriert für $n=2$ das lineare

**Beispiel 1.** Die Iterationsfolge

$$x_{i+1} = A\,x_i \quad \text{mit} \quad A = \begin{pmatrix} -0{,}8 & -0{,}4 \\ 0{,}5 & -0{,}5 \end{pmatrix}$$

ist für jeden Startwert $x_0 \in \mathbb{R}^2$ (linear) gegen den einzigen Fixpunkt (0, 0) konvergent, da für die Norm $\|A\| = \max\limits_{\|x\|=1} \|A\,x\| = \sqrt{0{,}9} < 1$ gilt.