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Some hidden harmonies between new and old geometric loci*

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1 Introduction

Geometric loci have always fascinated mathematicians. Many of the most beautiful and famous curves represent many geometric loci which have given a remarkable push to the creation of most of the mathematics we know. At once a geometric locus is determined, mathematicians often wonder if it can also be derived in a different way, because, if it is possible, the geometric locus gains a deeper meaning. We consider the map $H(u, v) \mapsto$

*Dedicated to the memory of the unforgettable friend and Master Prof. Filippo Spagnolo

Die Geschichte der Mathematik ist gespickt mit Entdeckungen von immer neuen geometrischen Örtern. Darunter sind solche, welche der Untersuchung bekannter Fragestellungen entsprangen, etwa den drei klassischen durch Zirkel und Lineal unlösbaren geometrischen Problemen der griechischen Antike. Bekannte Beispiele sind die Archimedische Spirale, die Konchoide von Nikomedes, die Zissoide von Diokles oder die Quadratrix des Hippias. Während Jahrhunderten blieben geometrische Örter beliebte Studienobjekte, und seit dem XVII. Jahrhundert erlauben die Werkzeuge der analytischen Geometrie und der Analysis die Entdeckung neuer oder die neue Interpretation altbekannter geometrischer Örter. Mit jeder neuen Methode oder Fragestellung ergeben sich auch heute noch immer wieder erhellende Zusammenhänge, so auch im vorliegenden Beitrag.

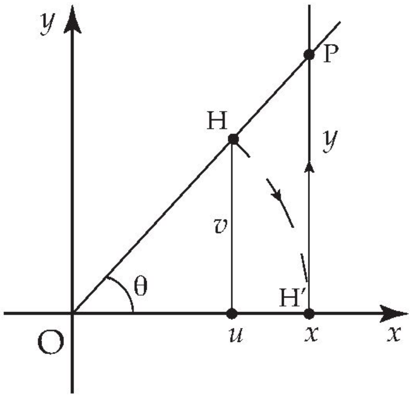

Fig. 1 The map $H(u, v) \mapsto P(x, y)$.

$P(x, y)$, a composition of a rotation of an angle θ of a point $H(u, v)$ upon the x-axis and a vertical projection of H' up to the point $P(x, y)$ (Figure 1), given (by Pythagoras and Thales) by the relations

$$\begin{cases} x &= \sqrt{u^2 + v^2} \\ y &= \dfrac{v}{u}x = \dfrac{v}{u}\sqrt{u^2 + v^2}. \end{cases} \tag{1}$$

In this paper we aim to show how, by this composition of a rotation and a vertical projection, it is possible to get already-known results, which sheds a new light on geometric objects.

2 The locus

Let r be a straight line referred to a cartesian coordinate system with $A(a, 0)$, $B(0, b)$ and consider the line t through the origin O of equation $y = mx$ (Figure 2). Let H be the point of intersection between r and t. Since the equation of the straight line through A and B is $bx + ay - ab = 0$, the point H has the coordinates:

$$\begin{cases} u = \dfrac{ab}{ma + b} \\ v = \dfrac{mab}{ma + b}. \end{cases}$$

Now if we apply the map (1) to the point H, we obtain the point P whose coordinates are:

$$\begin{cases} x = \dfrac{ab\sqrt{1 + m^2}}{ma + b} \\ y = \dfrac{mab\sqrt{1 + m^2}}{ma + b}. \end{cases} \tag{2}$$

Applying map (1) to the point H' of intersection between t and the straight line s parallel to r and passing through $A' = (-a, 0)$ and $B' = (0, -b)$ we obtain the point P'. Our

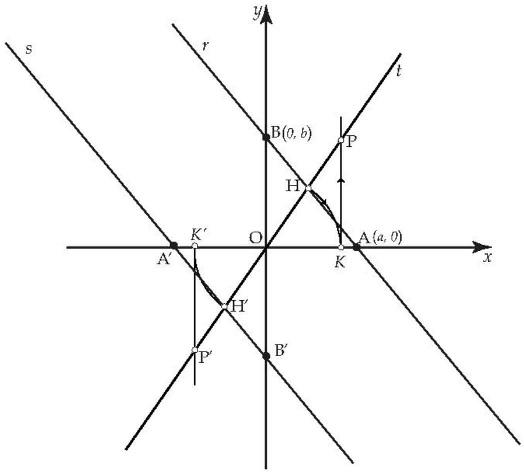

Fig. 2 The generation of the locus.

purpose is to study the geometric locus described by P and P' varying the straight line t. By symmetry we can study the case of point P. Since $m = y/x$, substituting into the first or second one of (2) we obtain

$$axy + bx^2 = ab\sqrt{x^2 + y^2}$$

from which, squaring, one has the quartic

$$b^2x^4 + 2abx^3y + a^2x^2y^2 - a^2b^2x^2 - a^2b^2y^2 = 0. \tag{3}$$

It is symmetric to the point O which is an isolated double point. It has (see Figure 3) two vertical asymptotes whose equations are $x = \pm b$, while the oblique asymptotes are:

$$y = -\frac{b}{a}x \pm \frac{b\sqrt{a^2 + b^2}}{a}.$$

If in (3) we divide by a^2 and we take the limit as a goes to infinity we obtain the curve of equation

$$x^2y^2 - b^2x^2 - b^2y^2 = 0 \tag{4}$$

which is the equation of a famous quartic named for its form: *cross curve* (Figure 4). In general, the equation of the cross curve is

$$x^2y^2 - a^2x^2 - b^2y^2 = 0$$

which, for $a = b$, is said to be *equilateral*, and it assumes the form of (4). It is also represented by the form

$$\frac{a^2}{y^2} + \frac{b^2}{x^2} = 1.$$

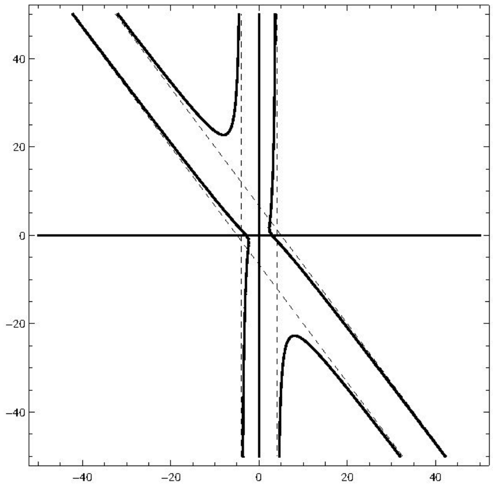

Fig. 3 The quartic (3) with $a = 3$ and $b = 4$.

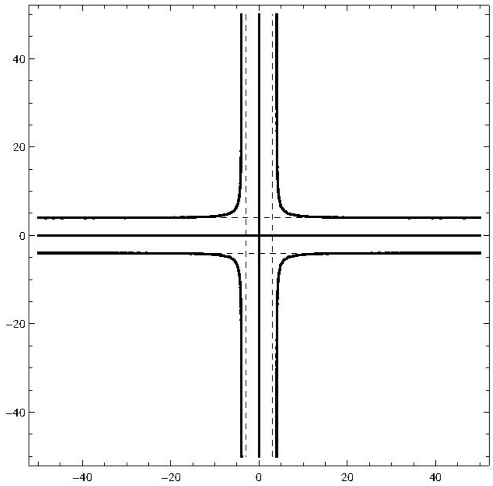

Fig. 4 The cross curve with $a = 3$ and $b = 4$.

The origin O is an isolated double point and the points at infinity on the axes are knots of inflection. The real points of the curve are external to the strips limited by the equations $x = \pm a$ and $y = \pm b$, which, in our case, (the equilateral one) are $x = \pm b$ and $y = \pm b$. This curve has been studied since 1847 in a question proposed by the French mathematician Olry Terquem in [3], and, thirteen years later, in a paper by the Italian mathematician Angelo Francesco Siacci, who demonstrated a fine property of the cross curve: *the sum of the areas limited by the four branches of the curve and by the respective asymptotes is equal to the area of the rectangle formed by the same asymptotes.* Siacci's proof of this

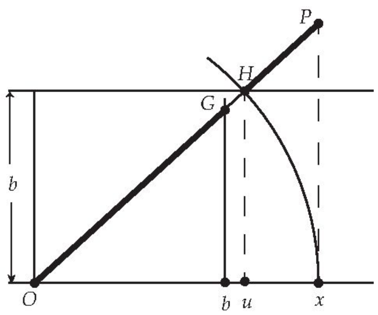

Fig. 5 The proof of the property of the Siacci cross curve.

property is sharp, [2] pp. 199–200, and makes use of calculus. Here we present a simpler geometric argument, brought to our attention by the anonymous referee whom we acknowledge. It is carried out as follows (Figure 5): multiply $x^2 - u^2 = b^2$ by the constant HO^2/u^2 to obtain, by Thales, $PO^2 - HO^2 = GO^2$. Hence, for any infinitesimally thin slice between two half-lines through O, the areas between PH and GO are the same. Therefore, the integrals are also the same.

The curve was also studied in [1] by the great Italian mathematician Ernesto Cesàro who named it *stauroid* from the Greek word $\sigma\tau\alpha\upsilon\rho o\varepsilon\iota\delta\eta\zeta$ which means crux-form, just for its form. Now if we divide (3) by b^2 and take the limit for $b \to \infty$ we obtain the quartic

$$x^4 - a^2x^2 - a^2y^2 = 0 \tag{5}$$

whose graph is illustrated in Figure 6.

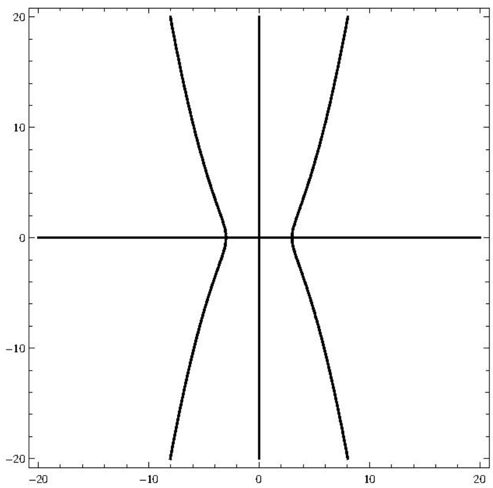

Fig. 6 The *kampyle* curve with $a = 3$.

The curve intersects the x-axis at $(-a, 0)$ and $(a, 0)$ and it is the famous kampyle curve $(\chi\alpha\mu\pi\upsilon\lambda\eta\gamma\rho\alpha\mu\mu\eta)$ which, according to Paul Tannery, was used by the great Greek mathematician Eudoxus of Cnidus to perform on the plane the construction devised by his teacher Archytas of Tarentum in order to solve the famous problem *duplication of cube,* known as the *Delian problem.* We recall the statement of the problem: given the edge of a cube, construct with *compasses and a straight edge alone the edge of a second cube having double the volume of the first.* Archytas of Tarentum devised a solution without compasses and a straight edge which is briefly as follows: his construction is not on a plane but a bold construction in three dimensions, determining a certain point as the intersection of three surfaces of revolution – a right cone, a cylinder, and a torus. The intersection of the two latter surfaces gives (says Archytas) a certain curve (which is, in fact, a curve of double curvature), and the point required is found where the cone meets this curve. Eudoxus of Cnidus was searching for a way to perform the construction devised by Archytas on a plane, and he obtained the kampyle curve.

We can obtain the kampyle curve using the map (1) again. In fact, let us consider two parallel straight lines r and s of equations $x = a$ and $x = -a$ respectively, and a straight line $y = mx$. Let us apply map (1) to the points $H(u, v)$ and $H'(-u, -v)$ (Figure 7).

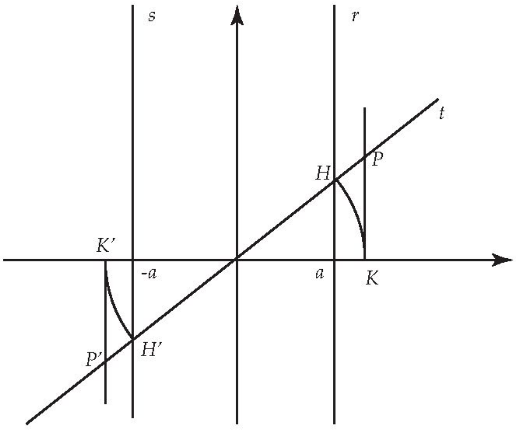

Fig. 7 A new generation of the *kampyle* curve.

By symmetry we consider only the point H, so for the coordinates of P one has:

$$\begin{cases} x = a\sqrt{m^2+1} \\ y = ma\sqrt{m^2+1}. \end{cases} \tag{6}$$

Substituting $y = mx$ into (6) we obtain

$$x^2 = a\sqrt{x^2+y^2}$$

so that by squaring we get the quartic

$$x^4 - a^2x^2 - a^2y^2 = 0$$

that is the equation of the *kampyle* curve.

The cross curve and the kampyle are reduced to the two isotropic straight lines arising from the origin both when one divides (4) by b^2 (with $b \to \infty$) and when we divide (5) by a^2 (with $a \to \infty$).

3 The case of the ellipse

Now we map (1) to the point $H(u, v)$ of intersection between the ellipse with semi-axes a and b of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the straight line r of the equation $y = mx$ and the point $P(x, y)$.

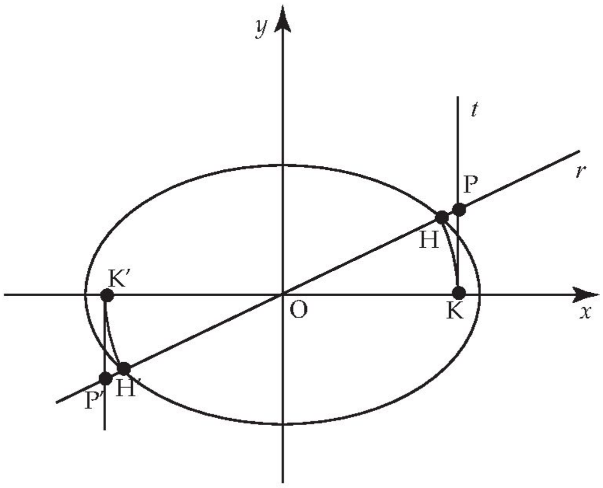

Fig. 8 The case of the ellipse.

Since:

$$\begin{cases} u = \dfrac{ab}{\sqrt{a^2m^2 + b^2}} \\ v = \dfrac{mab}{\sqrt{a^2m^2 + b^2}} \end{cases}$$

the action of map (1) gives

$$\begin{cases} x = ab\sqrt{\dfrac{1 + m^2}{a^2m^2 + b^2}} \\ y = mab\sqrt{\dfrac{1 + m^2}{a^2m^2 + b^2}}. \end{cases} \tag{7}$$

Since $y = mx$, substituting into the one of the two equations in (7) we obtain the quartic

$$b^2x^4 + a^2x^2y^2 - a^2b^2x^2 - a^2b^2y^2 = 0 \tag{8}$$

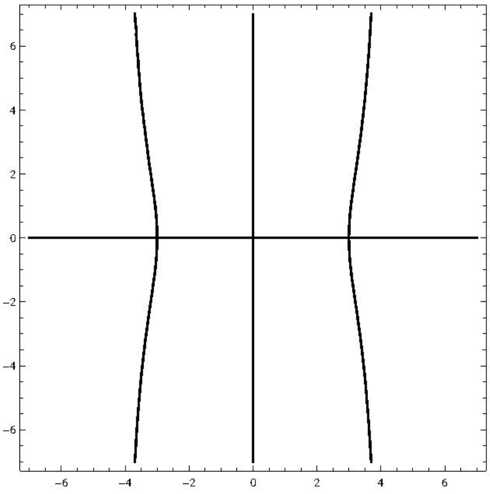

Fig. 9 The quartic (8) with $a = 3$ and $b = 4$.

which has, at the origin, an isolated double point with two vertical asymptotes of the general equation $x = \pm b$, whose graph is represented in Figure 9.
If in (8) we divide by a^2 and $a \to \infty$, we get the curve of equation

$$x^2y^2 - b^2x^2 - b^2y^2 = 0$$

that is, again, the equilateral cross curve. So, starting from simple geometric considerations, two famous curves historically obtained by other methods, both appear now as particular cases of general quartic curves. This offers a unitary vision and shows how often in mathematics subjects with no apparent linkage can find common significance with a mere change of the viewpoint.

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