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Limit shape of iterated Kiepert triangles

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We consider the following discrete dynamical system: start with a base triangle $\Delta_0 = ABC$ having at least two different vertices; erect externally on all sides of Δ_0 similar isosceles triangles with apex angle $\pi - 2\theta_0$; get the *Kiepert triangle* $\Delta_1 = A_1B_1C_1$ by taking the apices opposite to A, B, and C, respectively. Note that $\theta_0 = 0$ gives the medial triangle Δ_1, which is directly similar to Δ_0. We iterate the process with Δ_1 and $\pi - 2\theta_1$, Δ_2 and $\pi - 2\theta_2$, and so on. It is shown in [2] that this sequence of triangles has an equilateral limit shape when the apex angles are all equal and nonstraight. We give here a very short proof of this result and we determine the convergence behavior for any other choice of the apex angles.

Aus einem Dreieck kann ein neues Dreieck auf viele Arten entstehen. Ein Beispiel ist das Napoleon-Dreieck: Gleichschenklige Ohren mit einem Scheitelwinkel von 120°, die über den Seiten eines Dreiecks errichtet werden, ergeben immer ein gleichseitiges Dreieck. Oder die aufeinanderfolgenden Höhenfusspunktdreiecke eines Dreiecks: ihre Formenfolge kann beinahe jedes Verhalten aufweisen. Erwähnenswert ist auch die fraktale Struktur der Folge der Spiegelungsdreiecke, wo die Eckpunkte jeweils an der gegenüberliegenden Seite gespiegelt werden (G. Nicollier, *Iterated Reflection Triangles*, Forum Geometricorum 12 (2012), 83–129). Der Themenkreis der iterierten Dreiecke (und Polygone) ist sehr fruchtbar und bedient sich oft zyklischer Matrizen und der diskreten Fourier-Transformation. Die Autoren des vorliegenden Artikels verallgemeinern mit Hilfe einer Formfunktion einen Teil der Arbeit *Iterative Triangle Transformations* von D. Vartziotis und S. Huggenberger, die hier vor kurzem erschienen ist.

We use for any triangle $\Delta = UVW$ in the complex plane (with at least two different vertices) the shape function [1]

$$\sigma(\Delta) = \frac{U + V\zeta + W\zeta^2}{U + V\zeta^2 + W\zeta} \quad \text{with} \quad \zeta = e^{2\pi i/3}.$$

One has $\sigma(\Delta) = \sigma(\Delta')$ if and only if Δ and Δ' are directly similar for the given ordered vertices, and $\sigma(\Delta) = 0$ or ∞ means that Δ is a positively or negatively oriented equilateral triangle, respectively. The image of σ is the extended complex plane: one has $\sigma(\Delta) = \zeta$ for the vertices 0, 0, 1 in order, and $\sigma(\Delta) = s \neq \zeta$ for the vertices 0, 1, $\frac{\zeta s - 1}{\zeta - s}$.

Theorem 1. *If the successive angles $\theta_0, \theta_1, \dots \in \left]0, \frac{\pi}{2}\right[$ are bounded away from 0 and from $\frac{\pi}{2}$, the above sequence of triangles converges to an equilateral limit shape.*

Proof. Suppose that $\Delta_0 = ABC$ is positively oriented. The vertices of Δ_1 are then

$$C_1 = \frac{A + B}{2} + i\frac{A - B}{2}\tan\theta_0,$$

and cyclically. A simple computation gives

$$\sigma(\Delta_1) = \frac{1 - \sqrt{3}\tan\theta_0}{1 + \sqrt{3}\tan\theta_0} \cdot \sigma(\Delta_0)$$

(note that $\theta_0 = 30°$ proves Napoleon's theorem). Since Δ_1 is also positively oriented, one gets

$$\sigma(\Delta_2) = \frac{1 - \sqrt{3}\tan\theta_1}{1 + \sqrt{3}\tan\theta_1} \cdot \frac{1 - \sqrt{3}\tan\theta_0}{1 + \sqrt{3}\tan\theta_0} \cdot \sigma(\Delta_0),$$

and so on. Since each factor $\left|\frac{1-\sqrt{3}\tan\theta}{1+\sqrt{3}\tan\theta}\right|$ is smaller than some fixed bound $b < 1$ (Figure 1), one has $\lim_{n\to\infty} \sigma(\Delta_n) = 0$. □

Theorem 2. *If one excepts the trivial case of an equilateral base triangle Δ_0, the above sequence of iterated Kiepert triangles converges to an equilateral limit shape if and only if $\lim_{n\to\infty} \prod_{k=0}^{n} \frac{1-\sqrt{3}\tan\theta_k}{1+\sqrt{3}\tan\theta_k} = 0$, i.e., if and only if the successive angles $\theta_0 = \frac{\pi}{2}x_0$, $\theta_1 = \frac{\pi}{2}x_1, \dots \in \left[0, \frac{\pi}{2}\right[$ are such that $\theta_k = \frac{\pi}{6}$ for some k or $\sum_{k=0}^{\infty} \|x_k\| = \infty$, where $\|x\| = \min(x, 1 - x)$ denotes the distance from $x \in [0, 1]$ to the nearest integer 0 or 1.*

If the sequence of iterated Kiepert triangles has no equilateral limit shape (and if Δ_0 is positively oriented), the sequence $(\sigma(\Delta_n))$ has the nonzero limit

$$\sigma(\Delta_0) \prod_{k=0}^{\infty} \frac{1 - \sqrt{3}\tan\theta_k}{1 + \sqrt{3}\tan\theta_k}$$

when $\lim_{k\to\infty} \theta_k = 0$, and has otherwise exactly two accumulation points given by

$$\pm\sigma(\Delta_0) \prod_{k=0}^{\infty} \left|\frac{1 - \sqrt{3}\tan\theta_k}{1 + \sqrt{3}\tan\theta_k}\right|.$$

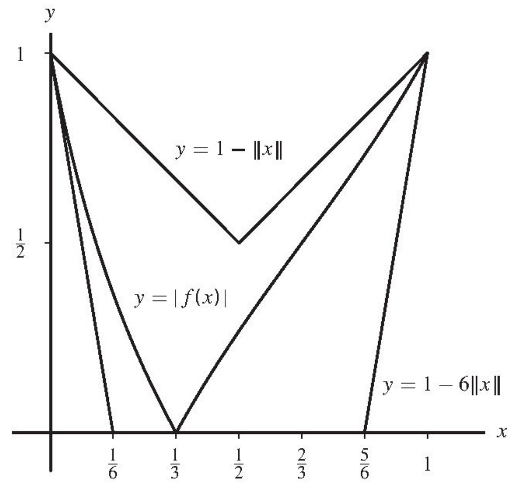

Figure 1

Proof. After continuous extension at $x = 1$, the function

$$f(x) = \frac{1 - \sqrt{3}\tan(\frac{\pi}{2}x)}{1 + \sqrt{3}\tan(\frac{\pi}{2}x)} = \frac{2}{1 + \sqrt{3}\tan(\frac{\pi}{2}x)} - 1$$

falls from 1 to -1 on $[0, 1]$ with $f(\frac{2}{3}) = -\frac{1}{2}$ (Figure 1), and its derivative

$$f'(x) = \frac{-\sqrt{3}\,\pi}{4\sin^2\big(\frac{\pi}{2}(x + \frac{1}{3})\big)}$$

grows on $[0, \frac{2}{3}]$ from $-\sqrt{3}\,\pi \approx -5.44$ at $x = 0$ to $\frac{-\sqrt{3}}{4}\pi$ at $x = \frac{2}{3}$, with $f'(\frac{1}{3}) = \frac{-\sqrt{3}}{3}\pi$, before falling to $\frac{-\sqrt{3}}{3}\pi$ on $[\frac{2}{3}, 1]$. One has thus (Figure 1)

$$1 - 6\|x\| \le |f(x)| \le 1 - \|x\| \le e^{-\|x\|}, \ \forall x \in [0, 1],$$

and $\prod_{k=0}^{\infty}|f(x_k)| \le e^{-\sum_{k=0}^{\infty}\|x_k\|}$: the infinite product is 0 or diverges to 0 when $x_k = \frac{1}{3}$ for some k or $\sum_{k=0}^{\infty}\|x_k\| = \infty$.

Conversely, suppose that all x_k are $\neq \frac{1}{3}$ and that $\sum_{k=0}^{\infty}\|x_k\| < \infty$: one has then $\|x_k\| < \frac{1}{10}$ as soon as k is large enough, say for all $k \ge K$, and one gets $\prod_{k=0}^{K-1}|f(x_k)| \neq 0$; further, since

$$1 - 6\|x\| \ge e^{-10\|x\|} \text{ for } \|x\| \le \tfrac{1}{10},$$

one obtains $\prod_{k=K}^{\infty}|f(x_k)| \ge e^{-10\sum_{k=K}^{\infty}\|x_k\|} > 0$. The full product $\prod_{k=0}^{\infty}|f(x_k)|$, whose factors lie in $]0, 1]$, has thus a limit $\lambda > 0$ (and any $\lambda \in]0, 1]$ can be obtained by an appropriate choice of the x_k's): by taking Δ_0 nonequilateral and positively oriented, $\sigma(\Delta_n) = \sigma(\Delta_0)\prod_{k=0}^{n-1} f(x_k)$ has a limit $\pm\lambda\sigma(\Delta_0) \neq 0$ if all $f(x_k)$ are eventually positive; otherwise, the infinite subsequences of the $\sigma(\Delta_n)$ with positive and negative $\prod_{k=0}^{n-1} f(x_k)$ have opposite limits $\lambda\sigma(\Delta_0)$ and $-\lambda\sigma(\Delta_0)$, respectively. Note that $\pm\lambda\sigma(\Delta_0)$ are, when $\neq -\sigma(\Delta_0)$, shapes of Kiepert triangles of Δ_0. □

References

[1] Hajja, M.; Martini, H.; Spirova, M.: On converses of Napoleon's theorem and a modified shape function. *Beitr. Algebra Geom.* 47 (2006), 363–383.

[2] Vartziotis, D.; Huggenberger, S.: Iterative geometric triangle transformations. *Elem. Math.* 67 (2012), 68–83.

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