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Autor: Bäsel, Uwe
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Buffon's problem with a pivot needle

Uwe Bäsel

Uwe Bäsel is professor for machine elements and mechanism theory at the Faculty of Mechanical and Energy Engineering at the Hochschule für Technik, Wirtschaft und Kultur (HTWK) Leipzig, with interests in applied mathematics, especially geometry and kinematics.

The classical Buffon needle problem asks for the probability that a needle of length ℓ thrown at random onto a plane lattice $\mathcal{R}_d$ of parallel lines at a distance $d \geq \ell$ apart will hit one of these lines. This problem was stated and solved by Buffon in his *Essai d'Arithmétique Morale*, 1777 (see, e.g., [5, pp. 71–72], [6, pp. 501–502]). If an arbitrary convex body $\mathcal{C}$ with maximum width $\leq d$ is used in this experiment, then the hitting probability is given by $u/(\pi d)$, where u denotes the perimeter of $\mathcal{C}$. This is the result of Barbier in 1860 [1, pp. 274–275], [6, p. 507]. If $\mathcal{C}$ is a needle (line segment), then $u = 2\ell$. If $\mathcal{C}$ is an ellipse, then there are elliptic integrals in the formulas of the hitting probabilities, see Duma and Stoka [3].

We consider a needle $\mathcal{N}_{a,b}$ consisting of two line segments $C'A'$, $C'B'$ of lengths $a := |C'A'|$ and $b := |C'B'|$, connected in a pivot point C' (see Fig. 1), and assume $a + b \leq d$.

Beim klassischen Buffonschen Problem wird eine Nadel betrachtet, die auf ein ebenes Gitter äquidistanter Geraden geworfen wird, wobei die Nadellänge maximal so groß wie der Abstand zwischen benachbarten Gittergeraden ist. Es wird nach der Wahrscheinlichkeit gefragt, dass die Nadel eine der Geraden trifft. Dieses Problem ist nach Georges-Louis Leclerc, Comte de Buffon (1707–1788) benannt, der es formulierte und 1777 in seinem *Essai d'Arithmétique Morale* löste. Heutzutage existiert eine große Anzahl von Arbeiten, die dieses Problem verallgemeinern. Wurfobjekt in der vorliegenden Arbeit ist eine Gelenknadel, die aus zwei gelenkig miteinander verbundenen Schenkeln besteht, wobei die Gesamtlänge der Nadel wiederum maximal so groß wie der Abstand zwischen benachbarten Gittergeraden sein soll. Eine derartige Nadel kann eine Gittergerade in einem oder zwei Punkten treffen, wofür die entsprechenden Wahrscheinlichkeiten berechnet werden. Diese Wahrscheinlichkeiten enthalten überraschenderweise das vollständige elliptische Integral zweiter Art, was sich im Fall verschieden langer Schenkel nicht weiter vereinfachen lässt.

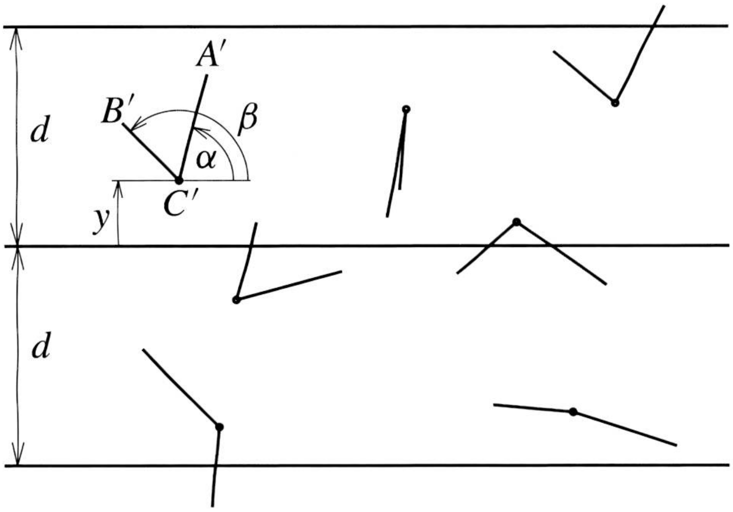

Figure 1 Lattice $\mathcal{R}_d$ and randomly thrown needle $\mathcal{N}_{a,b}$

The *random throw of* $\mathcal{N}_{a,b}$ *onto* $\mathcal{R}_d$ is defined as follows: The y-coordinate of the point C' is a random variable uniformly distributed in $[0, d]$. The angles α and β between the lines of $\mathcal{R}_d$, and segments $C'A'$ and $C'B'$, respectively, are random variables uniformly distributed in $[0, 2\pi]$. All three random variables are stochastically independent.

The probability of the event that $\mathcal{N}_{a,b}$ hits two lines of $\mathcal{R}_d$ at the same time is equal to zero, even in the case $a+b=d$. The expectation $\mathbb{E}(n)$ of the random variable $n =$ *number of intersection points between* $\mathcal{N}_{a,b}$ *and* $\mathcal{R}_d$ is given by $\mathbb{E}(n) = 2(a+b)/(\pi d)$, cf. [4].

Here we are asking for the probabilities $p(i)$, $i \in \{0, 1, 2\}$, of the events that $\mathcal{N}_{a,b}$ hits $\mathcal{R}_d$ in exactly i points. We denote by A and B the events that segments $C'A'$ and $C'B'$, respectively, hit one line of $\mathcal{R}_d$.

The following theorem states the result of this paper.

Theorem. *If* $a + b \leq d$, *then the probabilities* $p(i)$ *that* $\mathcal{N}_{a,b}$ *hits* $\mathcal{R}_d$ *in exactly i points are given by*

$$p(0) = 1 - \frac{(a+b)(\pi + 2E(k))}{\pi^2 d}, \quad p(1) = \frac{4(a+b)E(k)}{\pi^2 d},$$

$$p(2) = \frac{(a+b)(\pi - 2E(k))}{\pi^2 d},$$

where

$$E(k) = E(\pi/2, k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\theta}\, \mathrm{d}\theta$$

is the complete elliptic integral of the second kind with $k^2 = 4ab/(a+b)^2$.

Proof. We observe that the angle $\phi := \measuredangle(C'A', C'B')$ is a random variable uniformly distributed in $[0, 2\pi]$. Due to the result of Barbier, the conditional probability $P(A \cup B \mid \phi)$ of $A \cup B$ for fixed value of $\phi \in [0, 2\pi]$ is given by $u(\phi)/(\pi d)$, where $u(\phi)$ is the perimeter of the convex hull of $\mathcal{N}_{a,b}$. ($\mathcal{N}_{a,b}$ hits $\mathcal{R}_d$ if and only if its convex hull hits $\mathcal{R}_d$.) Using the law of total probability, the probability that $\mathcal{N}_{a,b}$ hits $\mathcal{R}_d$ is given by

$$\begin{aligned} P(A \cup B) &= \int_0^{2\pi} P(A \cup B \mid \phi)\, \frac{\mathrm{d}\phi}{2\pi} = \frac{1}{2\pi^2 d} \int_0^{2\pi} u(\phi)\, \mathrm{d}\phi \\ &= \frac{1}{2\pi^2 d} \int_0^{2\pi} \big[a + b + c(\phi)\big]\, \mathrm{d}\phi = \frac{a+b+\overline{c}}{\pi d}, \end{aligned}$$

where $c := |A'B'|$, and

$$\overline{c} := \frac{1}{2\pi} \int_0^{2\pi} c(\phi)\, \mathrm{d}\phi = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{a^2 + b^2 - 2ab\cos\phi}\, \mathrm{d}\phi\,.$$

Using $\cos\phi = 2\cos^2(\phi/2) - 1$, we have

$$\begin{aligned} \overline{c} &= \frac{1}{2\pi} \int_0^{2\pi} \sqrt{(a+b)^2 - 4ab\cos^2\frac{\phi}{2}}\, \mathrm{d}\phi \\ &= \frac{a+b}{2\pi} \int_0^{2\pi} \sqrt{1 - \frac{4ab}{(a+b)^2}\cos^2\frac{\phi}{2}}\, \mathrm{d}\phi\,. \end{aligned}$$

For abbreviation we put $k^2 = 4ab/(a+b)^2$. From the inequality $\sqrt{ab} \le (a+b)/2$ between the geometric and the arithmetic mean, one finds $k^2 \le 1$, hence $0 \le k \le 1$ with $k = 1$ only for $a = b$. With the substitution $\chi = \phi/2$ we get

$$\begin{aligned} \overline{c} &= \frac{a+b}{\pi} \int_0^{\pi} \sqrt{1 - k^2\cos^2\chi}\, \mathrm{d}\chi = \frac{2(a+b)}{\pi} \int_0^{\pi/2} \sqrt{1 - k^2\cos^2\chi}\, \mathrm{d}\chi \\ &= \frac{2(a+b)}{\pi} \int_0^{\pi/2} \sqrt{1 - k^2\sin^2\chi}\, \mathrm{d}\chi = \frac{2(a+b)E(k)}{\pi}\,. \end{aligned}$$

It follows that

$$\begin{aligned} P(A \cup B) &= \frac{a+b+\overline{c}}{\pi d} = \frac{(a+b)(\pi + 2E(k))}{\pi^2 d}, \\ P(A \cap B) &= P(A) + P(B) - P(A \cup B) = \frac{2a}{\pi d} + \frac{2b}{\pi d} - \frac{a+b+\overline{c}}{\pi d} \\ &= \frac{a+b-\overline{c}}{\pi d} = \frac{(a+b)(\pi - 2E(k))}{\pi^2 d}, \end{aligned}$$

and

$$\begin{aligned} p(0) &= 1 - P(A \cup B) = 1 - \frac{(a+b)(\pi + 2E(k))}{\pi^2 d}, \\ p(1) &= P(A \cup B) - P(A \cap B) = \frac{a+b+\overline{c}}{\pi d} - \frac{a+b-\overline{c}}{\pi d} = \frac{2\overline{c}}{\pi d} \\ &= \frac{4(a+b)E(k)}{\pi^2 d}, \\ p(2) &= P(A \cap B) = \frac{(a+b)(\pi - 2E(k))}{\pi^2 d}\,. \end{aligned}$$

□

This is the result from [2, pp. 57–58]. There it was obtained as special case of the more general result in Corollary 4.2 [2, p. 56].

Remark 1. If the angle ϕ is constant, then we have

$$P(A \cup B) = \frac{a+b+c}{\pi d} \quad \text{and} \quad P(A \cap B) = \frac{a+b-c}{\pi d}$$

with $c = \sqrt{a^2 + b^2 - 2ab\cos\phi}$. This yields

$$p(0) = 1 - \frac{a+b+c}{\pi d}, \quad p(1) = \frac{2c}{\pi d}, \quad p(2) = \frac{a+b-c}{\pi d},$$

see Santaló [5, pp. 77–78].

Remark 2. If $a = b$, we have $k = 1$, $E(1) = 1$, and therefore

$$p(0) = 1 - \frac{2a(\pi+2)}{\pi^2 d}, \quad p(1) = \frac{8a}{\pi^2 d}, \quad p(2) = \frac{2a(\pi-2)}{\pi^2 d}.$$

If $a \neq 0$ and $b = 0$, then $k = 0$ and $E(0) = \pi/2$, and therefore $P(A \cup B) = P(A) = 2a/(\pi d)$. This is the result of the classical Buffon needle problem.

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Uwe Bäsel
Hochschule für Technik, Wirtschaft
und Kultur (HTWK) Leipzig
Fakultät Maschinenbau und Energietechnik
PF 30 11 66
D-04251 Leipzig, Germany
e-mail: uwe.baesel@htwk-leipzig.de