

Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 72 (2017)
Heft: 3

Artikel: Parametric integrals, the Catalan numbers, and the beta function
Autor: Qi, Feng
DOI: <https://doi.org/10.5169/seals-730837>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 15.03.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Parametric integrals, the Catalan numbers, and the beta function

Feng Qi

Feng Qi is a full professor in mathematics at Henan Polytechnic University and Tianjin Polytechnic University. He received his PhD from the University of Science and Technology of China in 1999. Since 1993 he published nearly 500 papers. His current academic interests include special functions, computational number theory, and analytic combinatorics.

1 Introduction

Let a be a positive real number. For every non-negative integer n , Dana-Picard considered in [2] the definite integral

$$I_n(a) = \int_0^a x^n \sqrt{a^2 - x^2} \, dx \quad (1.1)$$

and inductively obtained that

$$I_0(a) = \frac{a^2\pi}{4}, \quad I_1(a) = \frac{a^3}{3}, \quad \text{and} \quad I_n(a) = \frac{a^2(n-1)}{n+2} I_{n-2}(a)$$

In der mathematischen Physik spielen *Spezielle Funktionen* eine tragende Rolle. Zu den am besten untersuchten Exemplaren dieser Funktionen gehören die Gammafunktion und die Eulersche Betafunktion, die sich durch die Eulerschen Integrale zweiter respektive erster Gattung ausdrücken lassen. Der Autor der vorliegenden Arbeit untersucht Familien von parametrischen Integralen und den Wallis-Quotienten $W_n = \frac{(2n-1)!!}{(2n)!!}$ und drückt deren Werte mit Hilfe der Gamma- und der Betafunktion aus. Dabei ergibt sich eine überraschende Verbindung zu den Catalan-Zahlen: Diese treten in der Kombinatorik in zahlreichen und auf den ersten Blick ganz verschiedenen Abzählproblemen auf. Die gefundenen Formeln erlauben dem Autor insbesondere den Nachweis, dass die Folge der Catalan-Zahlen absolut konvex ist.

for $n \geq 2$. Furthermore, by telescopic methods in [1, 3], Dana-Picard derived in [2] that

$$I_{2p}(a) = \left(\frac{a}{2}\right)^{2p+2} \frac{(2p)!}{p!(p+1)!} \pi, \quad p \in \mathbb{N} \quad (1.2)$$

and

$$I_{2p+1}(a) = \frac{a^{2p+3} 2^{2p+1}}{(2p+1)(2p+2)(2p+3)} \frac{p!(p+1)!}{(2p)!}, \quad p \geq 0. \quad (1.3)$$

Dana-Picard also observed in [2] that the quantities

$$C_p = \frac{(2p)!}{p!(p+1)!}, \quad p \geq 0$$

are just the Catalan numbers in combinatorics and that

$$C_p = \frac{1}{\pi} \int_0^2 x^{2p} \sqrt{4-x^2} \, dx, \quad p \in \mathbb{N}. \quad (1.4)$$

Dana-Picard further pointed out that the integral representation (1.4) is equivalent to

$$C_n = \frac{1}{2\pi} \int_0^4 x^n \sqrt{\frac{4-x}{x}} \, dx, \quad n \geq 0, \quad (1.5)$$

which was obtained in [11] by the Mellin transform. For more information on the Catalan numbers C_n , please refer to the monographs [6, 25] and the paper [18] and plenty of literature cited therein.

In this paper, we will present a unified expression of the formulas (1.2) and (1.3) in terms of the gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \quad \Re(z) > 0,$$

compute a new sequence of parametric integrals $\int_0^a \frac{x^n}{\sqrt{a^2-x^2}} \, dx$ for $a > 0$ and $n \geq 0$ in terms of the gamma function Γ , discover the absolute convexity of the Catalan numbers C_n , compute a general sequence of parametric integrals

$$I(a; \alpha, \beta) = \int_0^a x^\alpha (a^2 - x^2)^\beta \, dx \quad (1.6)$$

for $a > 0$ and $\alpha, \beta > -1$ in terms of the classical beta function

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \Re(x), \Re(y) > 0,$$

and represent the above sequences of parametric integrals $\int_0^a x^n (a^2 - x^2)^{\pm 1/2} \, dx$, the Catalan numbers C_n , and the Wallis ratio $W_n = \frac{(2n-1)!!}{(2n)!!}$ in terms of the classical beta function $B(x, y)$.

2 A unified expression of (1.2) and (1.3)

In this section, we present a unified expression of the formulas (1.2) and (1.3) in terms of the gamma function Γ as follows.

Theorem 2.1. *For $a > 0$ and $n \geq 0$, we have*

$$I_n(a) = a^{n+2} \frac{\sqrt{\pi} \Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{4\Gamma\left(\frac{n}{2} + 2\right)}. \quad (2.1)$$

Proof. By changing variables $x = a \sin s$ for $s \in [0, \frac{\pi}{2}]$, we have

$$\begin{aligned} I_n(a) &= a^{n+2} \int_0^{\pi/2} \sin^n s \sqrt{1 - \sin^2 s} \cos s \, ds \\ &= a^{n+2} \int_0^{\pi/2} \sin^n s \cos^2 s \, ds \\ &= a^{n+2} \left[\int_0^{\pi/2} \sin^n s \, ds - \int_0^{\pi/2} \sin^{n+2} s \, ds \right]. \end{aligned}$$

Since

$$\int_0^{\pi/2} \sin^n s \, ds = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}, \quad n \in \mathbb{N},$$

see [12, Section 1.1.3], it follows that

$$I_n(a) = a^{n+2} \left[\frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)} - \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+2}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2} + 1\right)} \right] = a^{n+2} \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2} + 2\right)}.$$

The proof of Theorem 2.1 is complete. \square

3 A new sequence of parametric integrals

Differentiating on both sides of the equation (2.1) produces a new sequence of parametric integrals $\int_0^a \frac{x^n}{\sqrt{a^2 - x^2}} \, dx$ and a new integral representation for the Catalan numbers C_n .

Theorem 3.1. *For $a > 0$ and $n \geq 0$, we have*

$$\int_0^a \frac{x^n}{\sqrt{a^2 - x^2}} \, dx = \sqrt{\pi} a^n \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)} \quad (3.1)$$

and, consequently,

$$C_n = \frac{2}{\pi(n+1)} \int_0^2 \frac{x^{2n}}{\sqrt{4 - x^2}} \, dx. \quad (3.2)$$

Proof. It is well known [16, Lemma 2.1] that

$$\frac{d}{dt} \int_{x_0}^t f(x, t) dx = f(t, t) + \int_{x_0}^t \frac{\partial f(x, t)}{\partial t} dx,$$

where $f(x, t)$ is differentiable in t and continuous in (x, t) in some region of the (x, t) -plane. Hence, differentiating with respect to a on both sides of (1.1) gives

$$I'_n(a) = a \int_0^a \frac{x^n}{\sqrt{a^2 - x^2}} dx. \quad (3.3)$$

On the other hand, differentiating with respect to a on both sides of (2.1) results in

$$I'_n(a) = \frac{\sqrt{\pi}}{4} (n+2) a^{n+1} \frac{\Gamma(\frac{n}{2} + \frac{1}{2})}{\Gamma(\frac{n}{2} + 2)}. \quad (3.4)$$

Combining (3.3) with (3.4) and simplifying lead to the formula (3.1).

The formula (3.2) follows readily from combination of

$$C_n = \frac{4^n \Gamma(n + \frac{1}{2})}{\sqrt{\pi} \Gamma(n + 2)}, \quad n \geq 0 \quad (3.5)$$

in [6, p. 112, Eq. (5.5)] and (3.1). The proof of Theorem 3.1 is complete. \square

4 Convexity of the Catalan numbers

It is common knowledge that a sequence $\{\mu_n\}_0^\infty$ is said to be convex if the inequality

$$\mu_{n+1} \leq \frac{\mu_n + \mu_{n+2}}{2}$$

is valid for every $n \geq 0$. An infinitely differentiable function f on an interval I is called absolutely convex on I if $f^{(2k)}(x) \geq 0$ on I . See [9, p. 375, Definition 3], [13, p. 2731, Definition 4.5], [22, p. 617, Definition 3], or [23, p. 3356, Definition 3]. A sequence $\{\mu_n\}_0^\infty$ is said to be absolutely convex if its elements are non-negative and its successive differences satisfy $\Delta^{2k} \mu_n \geq 0$ for $n, k \geq 0$, where

$$\Delta^k \mu_n = \sum_{m=0}^k (-1)^m \binom{k}{m} \mu_{n+k-m}.$$

It is clear that an absolutely convex function (sequence) must be convex.

Utilizing the integral representations (1.4), (1.5), and (3.2), we can derive the absolute convexity of the sequences C_n and $(n+1)C_n$.

Theorem 4.1. *The sequences C_n and $(n+1)C_n$ for $n \geq 0$ are both (absolutely) convex.*

Proof. The absolute convexity of the sequence C_n follows from the integral representations (1.4) or (1.5) and the absolute convexity of the functions $\int_0^2 x^t \sqrt{4-x^2} dx$ and $\int_0^4 x^t \sqrt{\frac{4-x}{x}} dx$ with respect to t .

The absolute convexity of the sequence $(n+1)C_n$ follows from rewriting the formula (3.2) as

$$(n+1)C_n = \frac{2}{\pi} \int_0^2 \frac{x^{2n}}{\sqrt{4-x^2}} dx$$

and the absolute convexity of the function $\int_0^2 \frac{x^t}{\sqrt{4-x^2}} dx$ with respect to t . The proof of Theorem 4.1 is complete. \square

5 A general sequence of parametric integrals

Motivated by the proof of Theorem 2.1, we now compute a general sequence of parametric integrals (1.6) in terms of the beta function $B(x, y)$.

Theorem 5.1. *For $a > 0$ and $\alpha, \beta > -1$, we have*

$$I(a; \alpha, \beta) = \int_0^a x^\alpha (a^2 - x^2)^\beta dx = \frac{1}{2} a^{\alpha+2\beta+1} B\left(\frac{\alpha+1}{2}, \beta+1\right). \quad (5.1)$$

Proof. By changing variables $x = a \sin t$ for $t \in [0, \frac{\pi}{2}]$ as in the proof of Theorem 2.1, we have

$$\begin{aligned} I(a; \alpha, \beta) &= \int_0^{\pi/2} (a \sin t)^\alpha [a^2 - (a \sin t)^2]^\beta a \cos t dt \\ &= a^{\alpha+2} \int_0^{\pi/2} \sin^\alpha t (1 - \sin^2 t)^\beta \cos t dt \\ &= a^{\alpha+2} \int_0^{\pi/2} \sin^\alpha t \cos^{2\beta+1} t dt = \frac{1}{2} a^{\alpha+2\beta+1} B\left(\frac{\alpha+1}{2}, \beta+1\right), \end{aligned}$$

where we used in the last step the formula

$$\int_0^{\pi/2} \sin^{2a-1} \theta \cos^{2b-1} \theta d\theta = \frac{1}{2} B(a, b), \quad \Re(a), \Re(b) > 0 \quad (5.2)$$

in [10, p. 142, Eq. 5.12.2]. The proof of Theorem 5.1 is complete. \square

6 Remarks

Making use of the formula (5.1), we now represent the sequences of parametric integrals $\int_0^a x^n (a^2 - x^2)^{\pm 1/2} dx$, the Catalan numbers C_n , and the Wallis ratio in terms of the classical beta function $B(x, y)$ in the form of remarks.

Remark 6.1. By the formula (5.1) in Theorem 5.1, the formula (2.1) can be rewritten in terms of the beta function $B(x, y)$ as

$$I_n(a) = \frac{1}{2} a^{n+2} B\left(\frac{n+1}{2}, \frac{3}{2}\right).$$

Remark 6.2. By virtue of (3.5) and Theorem 5.1, we immediately recover the relation

$$C_n = \frac{1}{\pi} I_{2n}(2) = \frac{1}{\pi} 2^{2n+1} B\left(\frac{2n+1}{2}, \frac{3}{2}\right) \quad (6.1)$$

and the integral representation (1.4) for $n \geq 0$.

Remark 6.3. By Theorem 5.1, the formula (3.1) can be rewritten in terms of the beta function $B(x, y)$ as

$$\int_0^a \frac{x^n}{\sqrt{a^2 - x^2}} dx = \frac{1}{2} a^n B\left(\frac{n+1}{2}, \frac{1}{2}\right).$$

Remark 6.4. It was stated in [5] that

$$\int_0^{\pi/2} \sin^t x dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{t+1}{2}\right)}{\Gamma\left(\frac{t+2}{2}\right)}, \quad t > -1. \quad (6.2)$$

See also [12, p. 16, Eq. (2.18)]. By (5.2), we can alternatively express the formula (6.2) in terms of the beta function $B(x, y)$ as

$$\int_0^{\pi/2} \sin^t x dx = \frac{1}{2} B\left(\frac{t+1}{2}, \frac{1}{2}\right).$$

Remark 6.5. It is well known that the Wallis ratio is defined by

$$W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+1/2)}{\Gamma(n+1)}, \quad n \in \mathbb{N}.$$

This quantity has been studied and applied by many mathematicians. See [4, 17, 19], for example, and plenty of literature therein. The Wallis ratio can be expressed in terms of the beta function $B(x, y)$ as

$$W_n = \frac{1}{\pi} B\left(n + \frac{1}{2}, \frac{1}{2}\right), \quad n \in \mathbb{N}.$$

Remark 6.6. Since

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt, \quad \Re(a), \Re(b) > 0,$$

see [10, p. 142, Eq. 5.12.1], the relation (6.1) can be rearranged as

$$\frac{C_n}{2^{2n}} = \frac{2}{\pi} \int_0^1 t^{n-1/2} (1-t)^{1/2} dt.$$

This implies that the sequence $\frac{C_n}{2^{2n}}$ for $n \geq 0$ is completely monotonic. For more information on the Catalan numbers C_n , their generalizations, and their (completely monotonic) properties, please refer to the monographs [6, 25], the formerly published papers [7, 8, 14, 15, 18, 20, 21, 24] and plenty of references therein.

Acknowledgement

The author is grateful to Dr. Thierry Dana-Picard at the Jerusalem College of Technology in Israel for his communicating and supplying the formally published versions of the papers [1, 2] through the ResearchGate.

The author is thankful to the anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

References

- [1] T. Dana-Picard, *Explicit closed forms for parametric integrals*, Internat. J. Math. Ed. Sci. Tech. **35** (2004), no. 3, 456–467; Available online at <http://dx.doi.org/10.1080/00207390410001686616>.
- [2] T. Dana-Picard, *Parametric integrals and Catalan numbers*, Internat. J. Math. Ed. Sci. Tech. **36** (2014), no. 4, 410–414; Available online at <http://dx.doi.org/10.1080/00207390412331321603>.
- [3] P. Glaister, *Factorial sums*, Internat. J. Math. Ed. Sci. Tech. **34** (2003), no. 2, 250–257; Available online at <http://dx.doi.org/10.1080/0020739031000158272>.
- [4] B.-N. Guo and F. Qi, *On the Wallis formula*, Internat. J. Anal. Appl. **8** (2015), no. 1, 30–38.
- [5] D.K. Kazarinoff, *On Wallis' formula*, Edinburgh Math. Notes **1956** (1956), no. 40, 19–21.
- [6] T. Koshy, *Catalan Numbers with Applications*, Oxford University Press, Oxford, 2009.
- [7] F.-F. Liu, X.-T. Shi, and F. Qi, *A logarithmically completely monotonic function involving the gamma function and originating from the Catalan numbers and function*, Glob. J. Math. Anal. **3** (2015), no. 4, 140–144; Available online at <http://dx.doi.org/10.14419/gjma.v3i4.5187>.
- [8] M. Mahmoud and F. Qi, *Three identities of the Catalan–Qi numbers*, Mathematics **4** (2016), no. 2, Article 35, 7 pages; Available online at <http://dx.doi.org/10.3390/math4020035>.
- [9] D.S. Mitrinović, J.E. Pečarić, and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993; Available online at <http://dx.doi.org/10.1007/978-94-017-1043-5>.
- [10] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, and C.W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010; Available online at <http://dlmf.nist.gov/>.
- [11] K. A. Penson and J.-M. Sixdeniers, *Integral representations of Catalan and related numbers*, J. Integer Seq. **4** (2001), no. 2, Article 01.2.5.
- [12] F. Qi, *Bounds for the ratio of two gamma functions*, J. Inequal. Appl. **2010** (2010), Article ID 493058, 84 pages; Available online at <http://dx.doi.org/10.1155/2010/493058>.
- [13] F. Qi, *Generalized weighted mean values with two parameters*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **454** (1998), no. 1978, 2723–2732; Available online at <http://dx.doi.org/10.1098/rspa.1998.0277>.
- [14] F. Qi and B.-N. Guo, *Logarithmically complete monotonicity of a function related to the Catalan–Qi function*, Acta Univ. Sapientiae Math. **8** (2016), no. 1, 93–102; Available online at <http://dx.doi.org/10.1515/ausm-2016-0006>.
- [15] F. Qi and B.-N. Guo, *Logarithmically complete monotonicity of Catalan–Qi function related to Catalan numbers*, Cogent Math. (2016), **3**:1179379, 6 pages; Available online at <http://dx.doi.org/10.1080/23311835.2016.1179379>.
- [16] F. Qi and L. Debnath, *Evaluation of a class of definite integrals*, Internat. J. Math. Ed. Sci. Tech. **32** (2001), no. 4, 629–633; Available online at <http://dx.doi.org/10.1080/00207390116734>.
- [17] F. Qi and M. Mahmoud, *Some properties of a function originating from geometric probability for pairs of hyperplanes intersecting with a convex body*, Math. Comput. Appl. **21** (2016), no. 3, Article 27, 6 pages; Available online at <http://dx.doi.org/10.3390/mca21030027>.

- [18] F. Qi, M. Mahmoud, X.-T. Shi, and F.-F. Liu, *Some properties of the Catalan–Qi function related to the Catalan numbers*, SpringerPlus (2016), **5**:1126, 20 pages; Available online at <http://dx.doi.org/10.1186/s40064-016-2793-1>.
- [19] F. Qi and C. Mortici, *Some best approximation formulas and inequalities for the Wallis ratio*, Appl. Math. Comput. **253** (2015), 363–368; Available online at <http://dx.doi.org/10.1016/j.amc.2014.12.039>.
- [20] F. Qi, X.-T. Shi, F.-F. Liu, and D.V. Kruchinin, *Several formulas for special values of the Bell polynomials of the second kind and applications*, ResearchGate Technical Report (2015), available online at <http://dx.doi.org/10.13140/RG.2.1.3230.1927>; J. Appl. Anal. Comput. (2017), in press.
- [21] F. Qi, X.-T. Shi, M. Mahmoud, and F.-F. Liu, *The Catalan numbers: a generalization, an exponential representation, and some properties*, ResearchGate Research (2015), available online at <http://dx.doi.org/10.13140/RG.2.1.1086.4486>; J. Comput. Anal. Appl. **22** (2017), in press.
- [22] F. Qi and S.-L. Xu, *Refinements and extensions of an inequality, II*, J. Math. Anal. Appl. **211** (1997), no. 2, 616–620; Available online at <http://dx.doi.org/10.1006/jmaa.1997.5318>.
- [23] F. Qi and S.-L. Xu, *The function $(b^x - a^x)/x$: inequalities and properties*, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3355–3359; Available online at <http://dx.doi.org/10.1090/S0002-9939-98-04442-6>.
- [24] X.-T. Shi, F.-F. Liu, and F. Qi, *An integral representation of the Catalan numbers*, Glob. J. Math. Anal. **3** (2015), no. 3, 130–133; Available online at <http://dx.doi.org/10.14419/gjma.v3i3.5055>.
- [25] R.P. Stanley, *Catalan Numbers*, Cambridge University Press, New York, 2015; Available online at <http://dx.doi.org/10.1017/CBO9781139871495>.

Feng Qi

Institute of Mathematics
Henan Polytechnic University
Jiaozuo City, Henan Province
454010, China

and

College of Mathematics
Inner Mongolia University for Nationalities
Tongliao City
Inner Mongolia Autonomous Region
028043, China
e-mails: qifeng618@gmail.com
qifeng618@hotmail.com
qifeng618@qq.com
URL: <https://qifeng618.wordpress.com>