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Autor: Hošek, Radim

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Short note

Expressing the remainder of the Taylor polynomial when the function lacks smoothness

Radim Hošek

Abstract. Taylor's theorem is a widely used tool for approximating a function by a polynomial. This is only possible when the function possesses continuous derivatives up to a corresponding order. To be able to get a more precise formula for the remainder, i.e., the difference between the function and its Taylor polynomial, the standard theorem requires the function to have in addition one more continuous derivative. In this paper we bring a simple generalization of such a result, allowing the highest-order derivative to have jump discontinuities.

1 Introduction

Taylor's theorem enables to approximate a sufficiently smooth function locally by polynomials. This is what makes it a very popular tool even far outside the mathematical community. Recently, a generalized version of the standard theorem appeared useful in our analysis of a numerical method for compressible flow [4].

Obviously, many extensions to Taylor's theorem have been done and published, let us name for example Díaz & Výborný [1, 2] or recent development in the direction of fractional derivatives, see, e.g., Liu et al. [5] among many others. Our aim is to introduce a generalization that requires weaker assumptions on the differentiability of the function, assuming only one-sided differentiability in its highest order. As a consequence, our Taylor-type theorem is well suited for functions with jumps in their highest derivative.

2 The Standard Taylor Theorem

We start with the standard definition.

Definition 2.1 (Taylor polynomial). *Let $k \in \mathbb{N} \cup \{0\}$ and $f \in C^k[a,b]$, then we define its* Taylor polynomial of the kth order *at $x_0 \in [a,b]$ by*

$$P_k(x_0; x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

We recall the version of Taylor's theorem that gives additional information about the *remainder term.*

Theorem 2.2 (Taylor's Theorem with Remainders). *Let $k \in \mathbb{N} \cup \{0\}$ and $f \in C^k[a,b] \cap C^{k+1}(a,b)$ and $P_k(x_0; .)$ be its Taylor polynomial of the order k at $x_0 \in [a,b]$. Then for any $x \in [a,b]$ there exist $\xi_L, \xi_C \in (x, x_0)$ (or $\in (x_0, x)$) such that*

$$f(x) - P_k(x) = \frac{f^{(k+1)}(\xi_L)}{(k+1)!}(x - x_0)^{k+1}, \tag{1}$$

$$f(x) - P_k(x) = \frac{f^{(k+1)}(\xi_C)}{k!}(x - \xi_C)^k (x - x_0). \tag{2}$$

The right-hand sides of (1) and (2) are called *Lagrange* and *Cauchy remainder forms*, respectively. Notice that for $k = 0$, (1) and (2) coincide and Theorem 2.2 reduces to the Mean Value Theorem.

The above claim does not tell how to find points ξ_L, ξ_C, nevertheless, one can still gain useful information. For instance, let $k = 1$ and f be convex, then from (1) we deduce

$$f(x) - P_1(x_0; x) \geq 0, \quad \text{for all } x \in [a,b], \tag{3}$$

since $f'' \geq 0$ in $[a,b]$. Clearly, (3) holds for any convex function f, not only C^2, as $P_1(x_0; .)$ is a tangent line to the graph of f at the point x_0. As such it lies under the graph of f, when f is convex.

This illustrates that when the assumptions of Theorem 2.2 are weakened, we still might make useful conclusions on the relation of the function and its Taylor polynomial. In particular, we will suppose the existence of *one-sided derivatives* of the functions.

3 Preliminaries

We define

$$f'_-(x) = \lim_{h \to 0^+} \frac{f(x) - f(x-h)}{h} \quad \text{and} \quad f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}, \tag{4}$$

if the above limits exist and call them *left (one-sided) derivative* of f at x, and *right (one-sided) derivative* of f at x, respectively.

Clearly, if $f'_-(x) = f'_+(x)$, then the derivative of f at x exists and $f'(x) = f'_-(x) = f'_+(x)$. We say that one-sided derivatives of f exist in (a,b), if they exist at every $x \in (a,b)$.

Next, $u \in C[a,b]$ having one-sided derivatives in (a,b) and $v \in C^1[a,b]$, comply with the following algebraic rules

$$(u+v)'_\pm = u'_\pm + v', \qquad (uv)'_\pm = u'_\pm v + uv'. \tag{5}$$

One-sided higher-order derivatives are defined in a natural way; choosing $f = g^{(k)}$ in (4) we can define the left- and right-sided $(k+1)$st derivative of g.

Further, we use the notation $\operatorname{co}\{x, y\} = [x, y]$ or $[y, x]$, in dependence on the ordering of points x, y.

Finally, one easily checks that for $I = (a, b)$, $r \in \mathbb{R}$ we have

$$x \in I \iff x + r \in I + r, \qquad x \in I \Rightarrow rx \in rI, \tag{6}$$

where we define $I + r := (a + r, b + r)$ and $rI := \text{co}\{ra, rb\}$. Notice that the backward implication in the latter claim of (6) holds, if we restrict ourselves to $r \in \mathbb{R} \setminus \{0\}$.

4 The Generalized Taylor Theorem

Having introduced all necessary notation allows us to state the main result.

Theorem 4.1 (Generalized Taylor Theorem). *Let $k \in \mathbb{N} \cup \{0\}$ and assume that $f \in C^k(\bar{I})$ has a one-sided $(k+1)$-st derivatives in I. Then for any $x_0 \in \bar{I}$, any $x \in \bar{I}$ there exist $\xi_L, \xi_C \in \text{int co}\{x, x_0\}$ such that we have*

$$f(x) - P_k(x_0; x) \in \frac{(x - x_0)^{k+1}}{(k+1)!} \cdot \text{co}\left\{ f_+^{(k+1)}(\xi_L), f_-^{(k+1)}(\xi_L) \right\}, \tag{7}$$

$$f(x) - P_k(x_0; x) \in \frac{(x - \xi_C)^k}{k!}(x - x_0) \cdot \text{co}\left\{ f_+^{(k+1)}(\xi_C), f_-^{(k+1)}(\xi_C) \right\}. \tag{8}$$

Theorem 4.1 is a Taylor type theorem for functions having jump discontinuities in their highest derivative. An application of Theorem 4.1 can be found in [4, Remark 3]. Notice that we could also include the case when the one-sided derivative diverges to $+\infty$ (or $-\infty$), the assertion remains unchanged.

The proof of Theorem 2.2 can be found for instance in [6, Section 5.3.3] and consists of the standard theorems of differential calculus. To prove Theorem 4.1 we follow the very same steps using our weaker assumptions on differentiability of the functions. Notice that one can recover the original proof from what we present simply by assuming that all functions are continuously differentiable of the appropriate order, which reduces all the intervals representing the possible jumps of derivatives to a single point, compare, e.g., (1) and (7).

We start with the following generalized version of Rolle's Theorem, which was also proved in [3].

Lemma 4.2 (Generalized Rolle Theorem). *Let $g \in C[a, b]$, $g(a) = g(b)$ and let the one-sided derivatives of g exist in (a, b). Then there exists $\xi \in (a, b)$ such that $0 \in \text{co}\{g'_+(\xi), g'_-(\xi)\}$.*

Proof. The continuity of g ensures (Weierstrass Theorem) that there exist its maximum $g(x_M) \geq g(a)$ and its minimum $g(x_m) \leq g(a)$. If g is a constant function, then ξ is any point of (a, b). If not, then at least one of the inequalities must be strict. Without loss of generality let us assume $g(x_M) > g(a) = g(b)$, hence x_M is in the interior of (a, b). To avoid contradiction with the maximality of $g(x_M)$, necessarily $g'_-(x_M) \geq 0$ and $g'_+(x_M) \leq 0$. In any case, this implies that $0 \in [g'_+(x_M), g'_-(x_M)]$, with this interval possibly degenerated to a single element set $\{0\}$. □

Lemma 4.3 (Generalized Cauchy Mean Value Theorem). *Let $v \in C^1[a,b]$ such that $v(a) \neq v(b)$, $u \in C[a,b]$ and let the one-sided derivatives of u exist in (a,b). Then there exists $\xi \in (a,b)$ such that*

$$\frac{u(b)-u(a)}{v(b)-v(a)}v'(\xi) \in \mathrm{co}\{u'_+(\xi), u'_-(\xi)\}. \tag{9}$$

Proof. Define the function w by

$$w(x) = u(x) - \frac{u(b)-u(a)}{v(b)-v(a)}v(x).$$

Then w is continuous in $[a,b]$ and its one-sided derivatives exist in (a,b), and one can check that $w(a) = w(b)$. The Generalized Rolle Theorem (Lemma 4.2) applied to w yields existence of some $\xi \in (a,b)$ such that

$$0 \in \mathrm{co}\{u'_+(\xi), u'_-(\xi)\} - \frac{u(b)-u(a)}{v(b)-v(a)}v'(\xi), \tag{10}$$

where we already used (6). Finally we rewrite (10) equivalently as (9). □

Now we can conclude with the proof of the central theorem.

Proof of Theorem 4.1. Let $k \geq 1$. Consider

$$F(t) := f(x) - P_k(t;x) = f(x) - \left[f(t) + f'(t)(x-t) + \cdots + \frac{f^{(k)}(t)}{k!}(x-t)^k \right].$$

Applying the algebra for one-sided derivatives (5) we compute

$$\begin{aligned} F'_\pm(t) &= -\left[f'(t) - f'(t) + f''(t)(x-t) - f''(t)(x-t) + \cdots + \frac{f_\pm^{(k+1)}(t)}{k!}(x-t)^k \right] \\ &= -\frac{f_\pm^{(k+1)}(t)}{k!}(x-t)^k. \end{aligned}$$

We use the Generalized Cauchy Theorem (Lemma 4.3) with the choice $u(t) = F(t)$, $v(t) = (x-t)^{k+1}$ and either $a = x_0, b = x$ or $a = x, b = x_0$ (depending on their ordering) to get

$$\frac{F(x)-F(x_0)}{-(x-x_0)^{k+1}}(-1)(k+1)(x-\xi)^k \in \mathrm{co}\left\{ -f_+^{(k+1)}(\xi)\frac{(x-\xi)^k}{k!}, -f_-^{(k+1)}(\xi)\frac{(x-\xi)^k}{k!} \right\}.$$

As $F(x) = 0$ and $F(x_0) = f(x) - P_k(x_0;x)$, we can use the algebra of intervals (6) to get

$$f(x) - P_k(x_0;x) \in \frac{(x-x_0)^{k+1}}{(k+1)!} \cdot \mathrm{co}\left\{ f_+^{(k+1)}(\xi), f_-^{(k+1)}(\xi) \right\},$$

which is (7). To prove (8), one just chooses $v(t) = x - t$ in the above consideration and performs the same steps. The case $k = 0$ (generalized Mean Value Theorem) is proved by virtue of Lemma 4.2 applied to the function

$$F(t) := h(t) - \frac{h(x) - h(x_0)}{x - x_0}(t - x_0), \tag{11}$$

or with possibly interchanged roles of x and x_0. □

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Radim Hošek
Institute of Mathematics
Czech Academy of Sciences
Zitna 25
115 67 Prague, Czech Republic
e-mail: hosek@math.cas.cz