

Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 72 (2017)
Heft: 4

Artikel: Short note : on the sums of Fibonacci and Lucas sequences or the art of cancelling $1-x$
Autor: Pauni, Dura
DOI: <https://doi.org/10.5169/seals-730843>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 15.03.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Short note **On the sums of Fibonacci and Lucas sequences or the art of cancelling $1 - x$**

Dura Paunić

Let $\{F_n : n \in \mathbb{N}_0\} = \{0, 1, 1, 2, 3, 5, \dots\}$, and $\{L_n : n \in \mathbb{N}_0\} = \{2, 1, 3, 4, 7, 11, \dots\}$, be sequences of Fibonacci, and Lucas numbers. Their generating functions are

$$F(x) = \frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n, \quad \text{and} \quad L(x) = \frac{2 - x}{1 - x - x^2} = \sum_{n=0}^{\infty} L_n x^n.$$

Recently, Sury [3], and Marques [2] established the Fibonacci–Lucas relations

$$\sum_{k=0}^n 2^k L_k = 2^{n+1} F_{n+1}, \quad \text{and} \quad \sum_{k=0}^n 3^k L_k + \sum_{k=0}^{n+1} 3^{k-1} F_k = 3^{n+1} F_{n+1}.$$

The first identity was elegantly proved by Kwong [1].

These two identities can be generalized easily using generating functions.

Let $G(x) = g_0 + g_1 x + g_2 x^2 + \dots + g_n x^n + \dots$, be any generating function. It follows that the coefficient of x^n is

- a) $m^n g_n$, for $G(mx)$, b) g_{n-1} for $xG(x)$, $n > 0$,
 c) g_{n+1} for $\frac{G(x)}{x}$, $n \geq 0$, d) $\sum_{k=0}^n g_k$, for $\frac{G(x)}{1-x} = \sum_{j=0}^{\infty} g_j x^j \times \sum_{k=0}^{\infty} x^k$.

Then for $m > 0$, $\sum_{k=0}^n m^k L_k + (m - 2) \sum_{k=0}^{n+1} m^{k-1} F_k$, is the coefficient of x^n in

$$\begin{aligned} & \frac{L(mx)}{1-x} + \frac{(m-2)F(mx)}{mx(1-x)} \\ &= \frac{2-mx}{(1-x)(1-(mx)-(mx)^2)} + \frac{(m-2)mx}{mx(1-x)(1-(mx)-(mx)^2)} \\ &= \frac{m}{1-(mx)-(mx)^2} = \frac{F(mx)}{x}. \end{aligned}$$

So, for nonnegative integer n , and real $m > 0$, the identity

$$\sum_{k=0}^n m^k L_k + (m - 2) \sum_{k=0}^{n+1} m^{k-1} F_k = m^{n+1} F_{n+1}$$

holds.

References

- [1] H. Kwong, An alternate proof of Sury's Fibonacci–Lucas relation, *Amer. Math. Monthly*, 121 (2014), 514.
- [2] D. Marques, A new Fibonacci–Lucas relation, *Amer. Math. Monthly*, 122 (2015), 683.
- [3] B. Sury, A polynomial parent to a Fibonacci–Lucas relation, *Amer. Math. Monthly*, 121 (2014), 236.

Đura Paunić

Department of Mathematics and Informatics

Faculty of Sciences

University of Novi Sad

Trg Dositeja Obradovića 4

21000 Novi Sad, Serbia

e-mail: djura@dmi.uns.ac.rs