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Short note

Gerretsen and the Finsler–Hadwiger inequality

Gerhard Wanner

The inequalities. The beautiful paper [2] (1953) by J.C.H. Gerretsen, written in Dutch language, is of difficult access[1]. It starts by stating Weitzenböck's inequality from 1919 [4]

$$a^2 + b^2 + c^2 \geq 4\mathcal{A}\sqrt{3}\,, \tag{1}$$

where $\mathcal{A}$ denotes the area of a triangle with sides a, b, c. Next follows its sharpening by P. Finsler and H. Hadwiger [1] from 1938

$$a^2 + b^2 + c^2 \geq 4\mathcal{A}\sqrt{3} + Q\,, \tag{2}$$

where

$$Q = (a-b)^2 + (b-c)^2 + (c-a)^2 \tag{3}$$

can be seen as a measure for the *"ongelijkzijdigheidsgraad"*[2] of the triangle. In both inequalities we have equality only for the equilateral triangle. Gerretsen then elaborates in Art. 2 the original proof of Finsler and Hadwiger ("Het bewijs ... is nogal listig") and gives simpler proofs for (1) and (2) in Arts. 3 and 4.

A further consequence. In Art. 5 Gerretsen continues as follows: From the Cosine Theorem

$$a^2 = b^2 + c^2 - 2bc\cos\alpha = b^2 + c^2 - 4\mathcal{A}\cot\alpha$$

we obtain by cyclic permutations and addition[3]

$$\Sigma\, a^2 = 4\mathcal{A}\,\Sigma\,\cot\alpha$$

[1]The author is grateful to W. Hundstorfer, Amsterdam, for providing him with a copy ("The scanner is not used often anymore, everything is on the internet. Well ... *almost* everything").

[2]literally "degree of unequilaterality"; Gerretsen adds: "as long as one does not find this word to crazy [wanneer men dit woord niet te gek vindt]".

[3]Gerretsen leaves the meaning of the notations $\Sigma\, a^2 = a^2 + b^2 + c^2$ or $\Sigma\,\cot\alpha = \cot\alpha + \cot\beta + \cot\gamma$ or $\Sigma\, ab = ab + bc + ca$ to the intelligence of the reader.

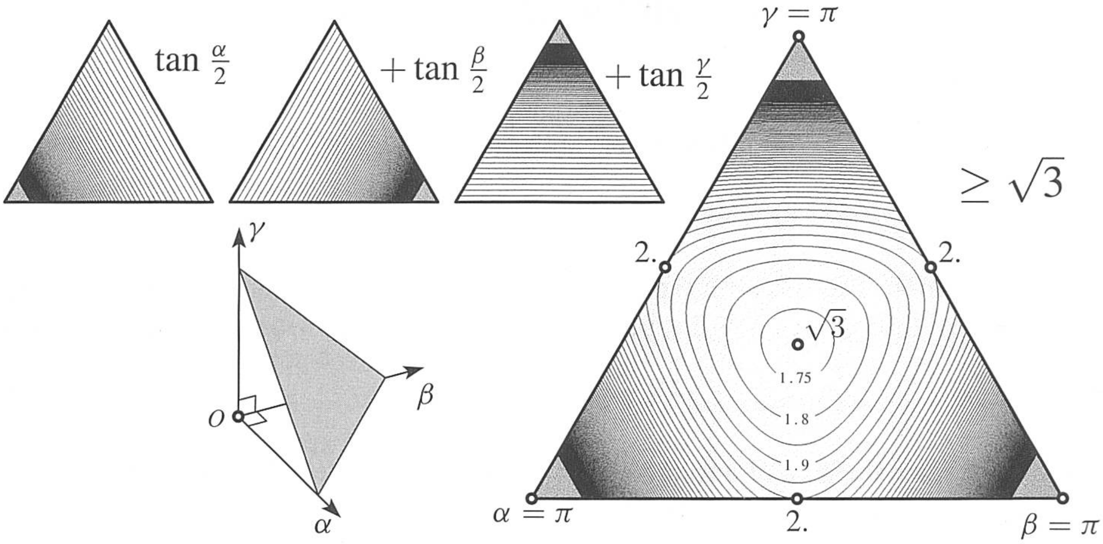
so that

$$\Sigma\, a^2 - Q = 2\,\Sigma\, ab - \Sigma\, a^2 = 4\mathcal{A}\,\Sigma\, \left(\frac{1}{\sin\alpha} - \cot\alpha\right) = 4\mathcal{A}\,\Sigma\, \tan\frac{\alpha}{2}\,, \tag{4}$$

thus the inequality of Finsler and Hadwiger means that

$$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} \geq \sqrt{3} \tag{5}$$

for all angles α, β, γ of a triangle.



Proof of $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} \geq \sqrt{3}$

Backward proof. However, we believe the inequality (5) at once: the angles α, β, γ of a triangle satisfying $\alpha + \beta + \gamma = \pi$ form in $\mathbb{R}^3$ an equilateral triangle for which (α, β, γ) are the barycentric coordinates (see the Figure above). The three terms of (5) are convex functions on this triangle, each one strictly convex in another direction, therefore their sum is strictly convex with a 120°-symmetry. Thus its minimal point, which must be unique, can only lie in the center where $\alpha = \beta = \gamma = \frac{\pi}{3}$ and the sum is $\sqrt{3}$. Reading now the proof in (4) backwards, we obtain another nice proof of the Finsler–Hadwiger inequality.

Epilog. In Art. 7 Gerretsen proves "his" famous inequalities with the standard proof from the Euler distances of the "bijzondere punkten" (see, e.g., [3]), and, inversely, Lukarevski [3] proves the Finsler–Hadwiger inequality from Gerritsen's inequalities.

References

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[4] R. Weitzenböck, *Über eine Ungleichung in der Dreiecksgeometrie*, Math. Zeitschr. 5 (1919) 137–146.

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