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Short note **Composite values of irreducible polynomials**

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In his letter to Euler from September 1743 in [3, Letter 73], Goldbach remarked that it

> *is very easy to prove that no algebraic formula such as* $a+bx+cx^2+dx^3+\cdots$*, where* x *is the index of the terms, can yield none but prime numbers, whatever integers the coefficients* a*,* b*,* c*, … may be; but all the same there are formulae which comprise a greater number of primes than many others; the series* $x^2+19x-19$ *is of this kind, as in its* 47 *initial terms it comprises only* 4 *non-prime numbers.*

In this note we will give a very short proof of Goldbach's claim based on a simple identity, which shows not only the existence of infinitely many composite values of a given polynomial, but presents identities from which the claim follows directly. As an example, applying our result to Goldbach's polynomial $f(x)=x^2+19x-19$ provides us with the identity

$$f(x^2+20x-19)=f(x)\cdot g(x) \quad \text{with} \quad g(x)=x^2+21x+1=f(x+1),$$

which implies that f attains infinitely many composite values. In particular, $f(25)=f(f(2)+2)=f(2)\cdot g(2)=23\cdot 47$.
Observe that Goldbach's claim is trivial if f is reducible or if its content (the greatest common divisor of its coefficients) is not a unit. Our main result is the following

Theorem 1. *Let* $f\in\mathbb{Z}[x]$ *be an irreducible polynomial with integral and coprime coefficients. Then for an arbitrarily chosen polynomial* $q(x)\in\mathbb{Z}[x]$ *there exists a polynomial* $g\in\mathbb{Z}[x]$ *such that*

$$f(q(x)f(x)+x)=f(x)g(x). \tag{1}$$

We have to show that for every choice of $q(x)\in\mathbb{Z}[x]$, the polynomial

$$h(x)=f(q(x)f(x)+x)$$

is divisible by $f(x)$. Since f is irreducible, a polynomial h is divisible by f in the ring $\mathbb{Q}[x]$ if and only if $h(\alpha)=0$ for all the complex roots α of f. But if $f(\alpha)=0$, then

$$h(\alpha)=f(q(\alpha)f(\alpha)+\alpha)=f(\alpha)=0,$$

and we are done. Gauss's Lemma for polynomials (see [4] and [2] for the history of this result) now tells us that if $h=fg$ for polynomials $f,h\in\mathbb{Z}[x]$ for a primitive polynomial f, then the coefficients of g must be integral. This completes the proof.

This implies in particular that polynomials f with degree ≥ 1 represent infinitely many composite numbers of the form $f(f(x)+x)$. In fact, assume that $f(x) = a_n x^n + \cdots + a_0$ with $a_n \geq 1$. Then there is a constant $C > 0$ such that $f(x) > 1$ and $f'(x) > 0$ for all $x > C$. But then $f(x) + x > x$, hence $f(f(x) + x) > f(x)$ and thus also $g(x) > 1$.

A second proof. Theorem 1 may also be proved by setting $h = q(x)f(x)$ in the Taylor identity

$$f(x+h) = f(x) + f'(x) \cdot h + \frac{f''(x)}{2!}h^2 + \cdots + \frac{f^{(n+1)}(x)}{(n+1)!}h^{n+1}.$$

This implies

$$f(x+f(x)) = f(x)\left[1+f'(x)q(x)+\frac{f''(x)}{2!}f(x)q(x)^2+\cdots+\frac{f^{(n+1)}(x)}{(n+1)!}f(x)^n q(x)^{n+1}\right].$$

Observe that the polynomials $\frac{1}{k!}f^{(k)}(x)$ have integral coefficients since the product of k consecutive integers is divisible by $k!$.

Applications to Goldbach's polynomial. Out of the four composite values of $f(n)$ for $0 \leq n \leq 47$, where $f(x) = x^2 + 19x - 19$ is Goldbach's polynomial, the numbers $f(19)$ and $f(38)$ (and more generally $f(19k)$ for integers $k \geq 1$) are composite for trivial reasons: they are clearly divisible by 19. The other two composite values are $f(25) = f(2 + f(2))$ and $f(36) = f(-f(-1) - 1)$. The next few composite values also follow from our theorem.

André Gérardin. I was led to the problem addressed here by a remark by André Gérardin (1879–1953) in [1], in which he claimed that the numbers of the form $2a^2-1$ are composite for $a = 9, 89, 881$ etc. I quickly found that these numbers are solutions of the diophantine equation $3a^2 - 2b^2 = 1$, and that Gérardin's claim follows from the observation that

$$9(2x^2 - 1) = 3(4y^2 - 1) = 3(2y - 1)(2y + 1).$$

This is reminiscent of the well-known fact that there are infinitely many composite integers of the form $f(x) = 4x^2 + 1$ since

$$f(a^2) = 4a^4 + 1 = (2a^2 + 1)^2 - 4a^2 = (2a^2 + 2a + 1)(2a^2 - 2a + 1).$$

This begs the question whether there is a result that encompasses all these examples.

References

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